1.3 — Sequential Games

ECON 316 • Game Theory • Fall 2021

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Outline

Games in Extensive Form

Mover Advantages

How Reasonable is Rollback Thinking?

The Century Mark Game

Rules: Two (teams of) players alternating turns

- The count starts at 0
  1. Team 1 adds a number 1-10 to the tally
  2. Team 2 adds a number 1-10 to the tally
- The first team to bring the tally to 100 wins

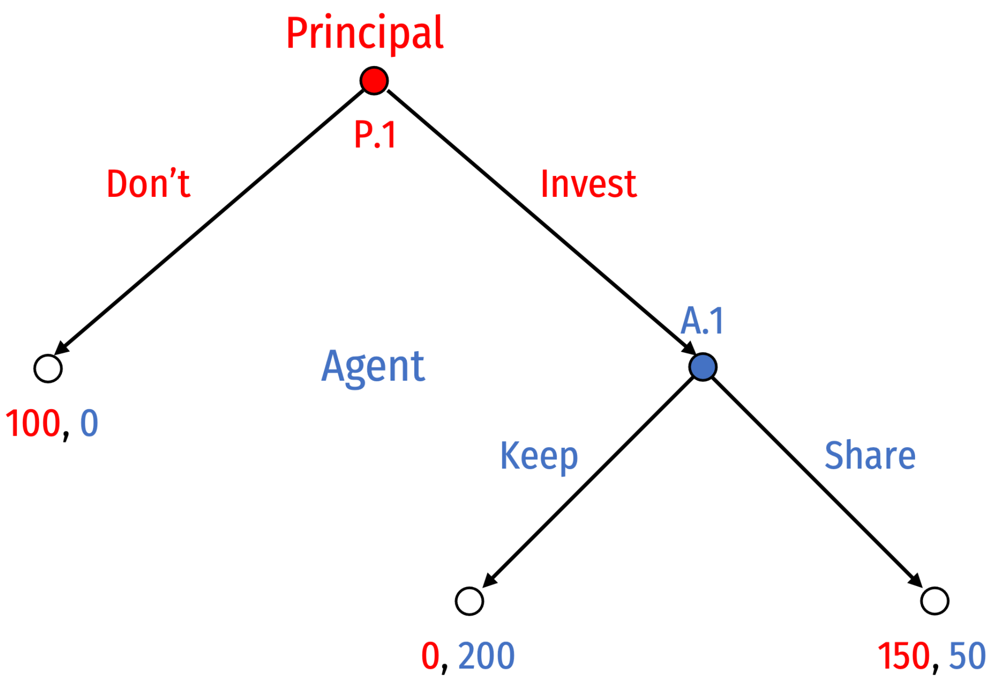
Sequential Games with Perfect Information

- Strict order of play
- Perfect information
  - No **external uncertainty**: nature/probability does not interfere between choices → outcomes
  - No **strategic uncertainty**: each player observes the history of other players' moves
- Can be represented in **extensive form**, i.e. a **game tree**

Games in Extensive Form

Games in Extensive Form

- **Example**: "trust" game
- **Principal** starts with \$100. If they invest, with **Agent**, it doubles to \$200
- **Agent** then decides whether to share or keep it



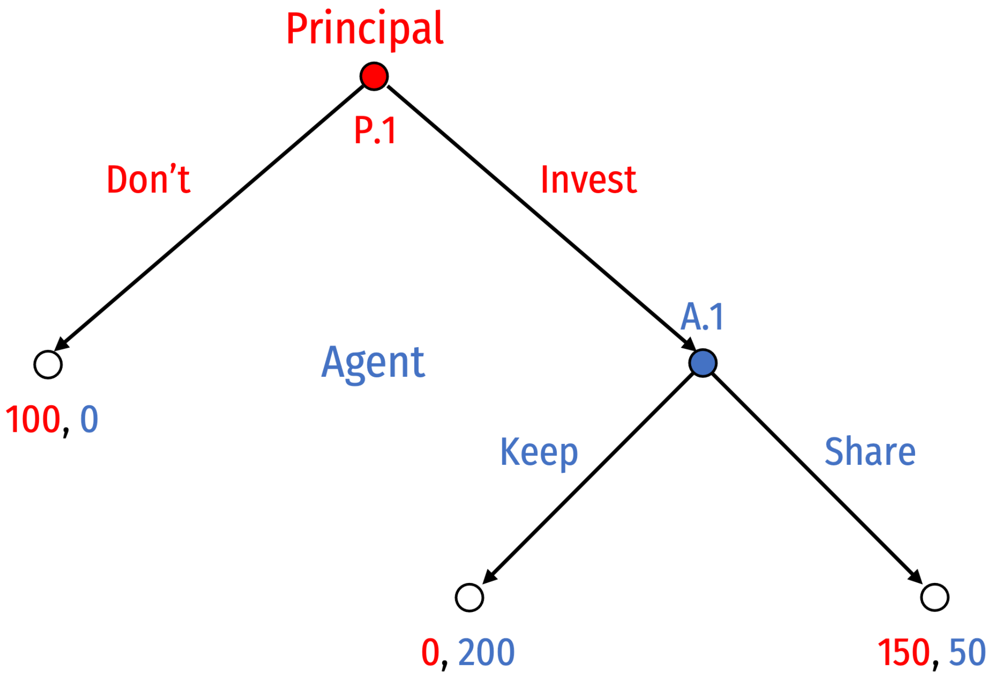
Games in Extensive Form

- Example: "trust" game

1. **Principal** (Player 1) moves first.

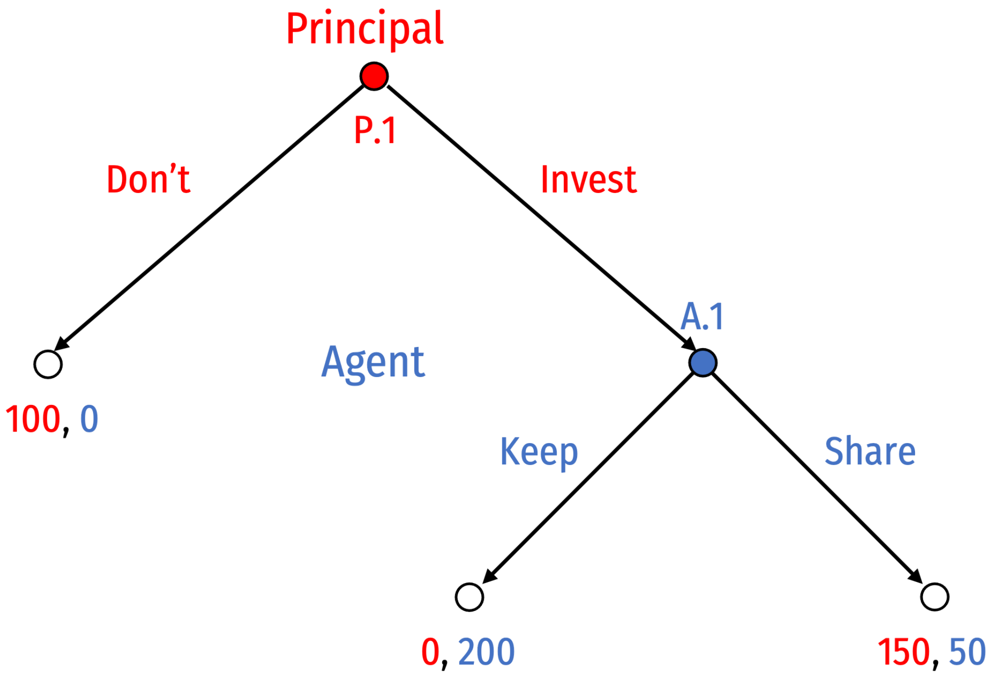
2. **Agent** (Player 2) moves second (but only if **Principal** has played Invest).

- The game ends.



Games in Extensive Form

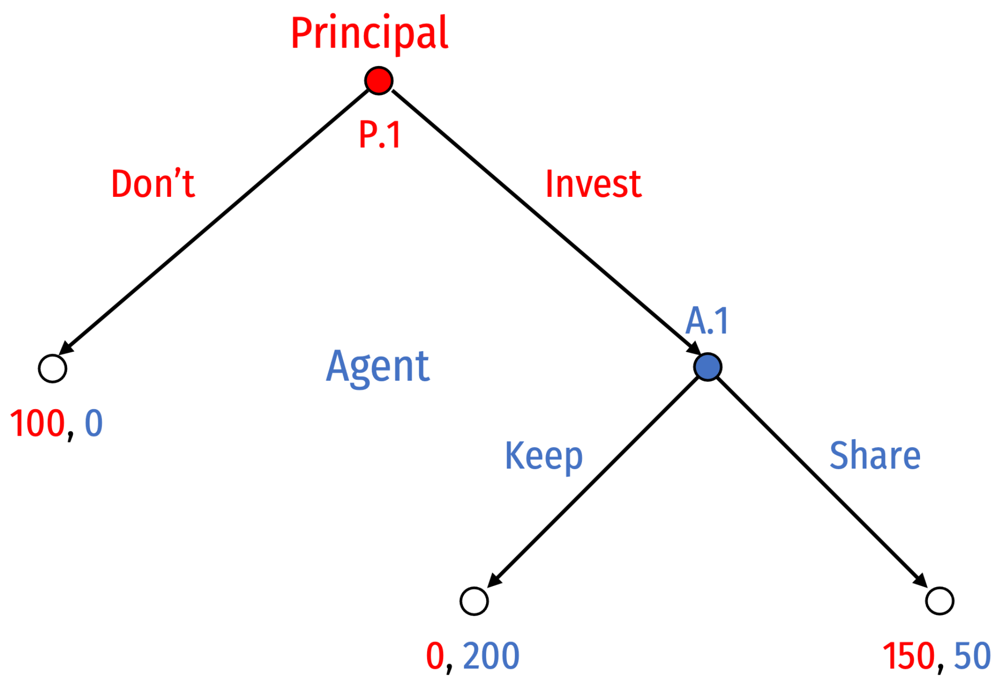
- Designing a game tree:
- **Decision nodes**: decision point for each player
  - Solid nodes, I've labeled and color-coded by player (P.1, A.1)
- **Terminal nodes**: outcome of game, with payoff for each player
  - Hollow nodes, no further choices



Games in Extensive Form: Outcomes

- Three possible outcomes:

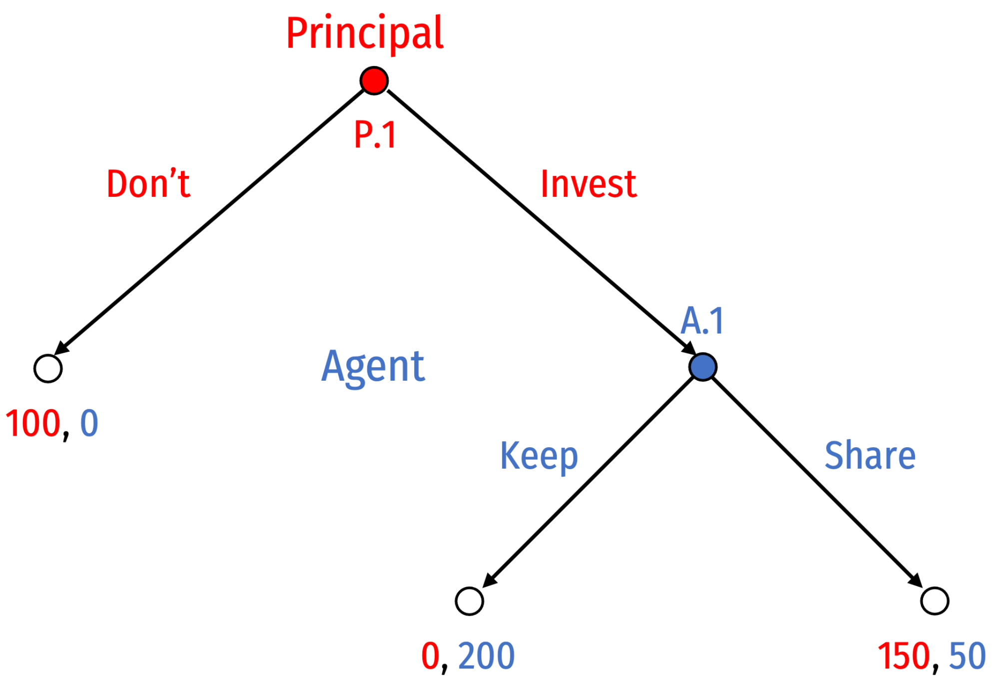
1. (Don't): 100, 0
2. (Invest, Keep): 0, 200
3. (Invest, Share): 150, 50



Strategies

- **("Pure") strategy**: a player's **complete** plan of action for every possible contingency
  - i.e. what player will choose at **every possible decision node**, even if it's never reached
- Think of a strategy like an **algorithm**:

> If we reach node 1, then I will play X; if we reach node 2, then I will play Y; if...



Trust Game: Strategies

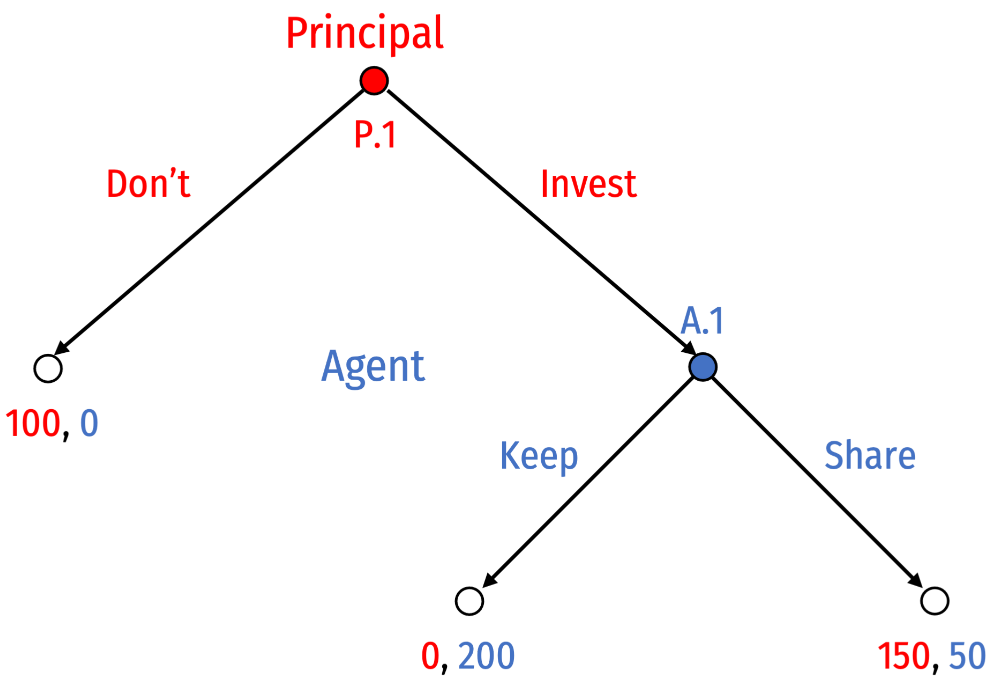
- **Principal** has 2 possible strategies:

    1. Don't at P.1
    2. Invest at P.1

- **Agent** has 2 possible strategies:

    1. Keep at A.1
    2. Share at A.1

- Note **Agent**'s strategy only comes into play if **Principal** plays Invest and the game reaches node A.1



Solving the Game: Backward Induction

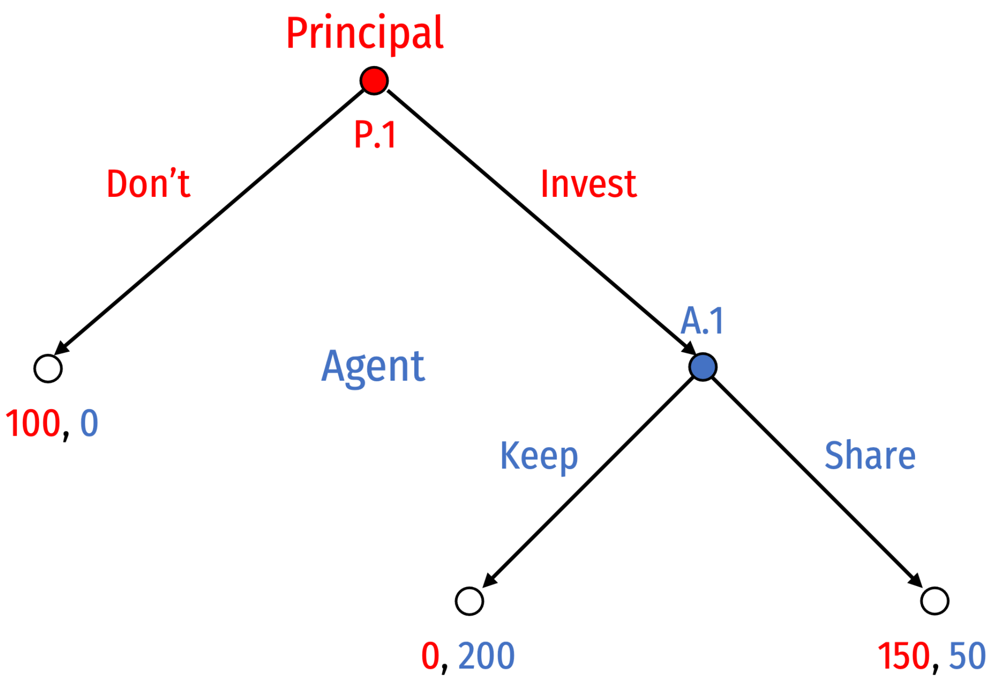
- Solve a sequential game by **“backward induction”** or **“rollback”**

- To determine the outcome of the game, start with the *last-mover* (i.e. decision nodes just before terminal nodes) and work to the beginning

- A process of considering **“sequential rationality”**:

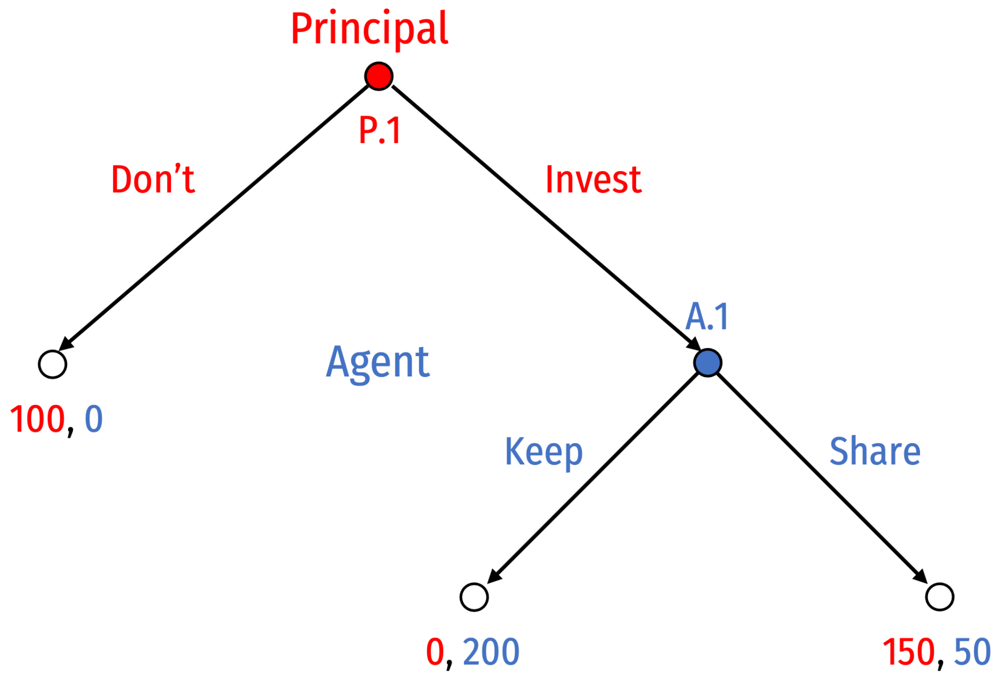
> “If I play X, my opponent will respond with Y; given their response, do I really want to play X?”

- What is that mover's best choice to maximize their payoff?



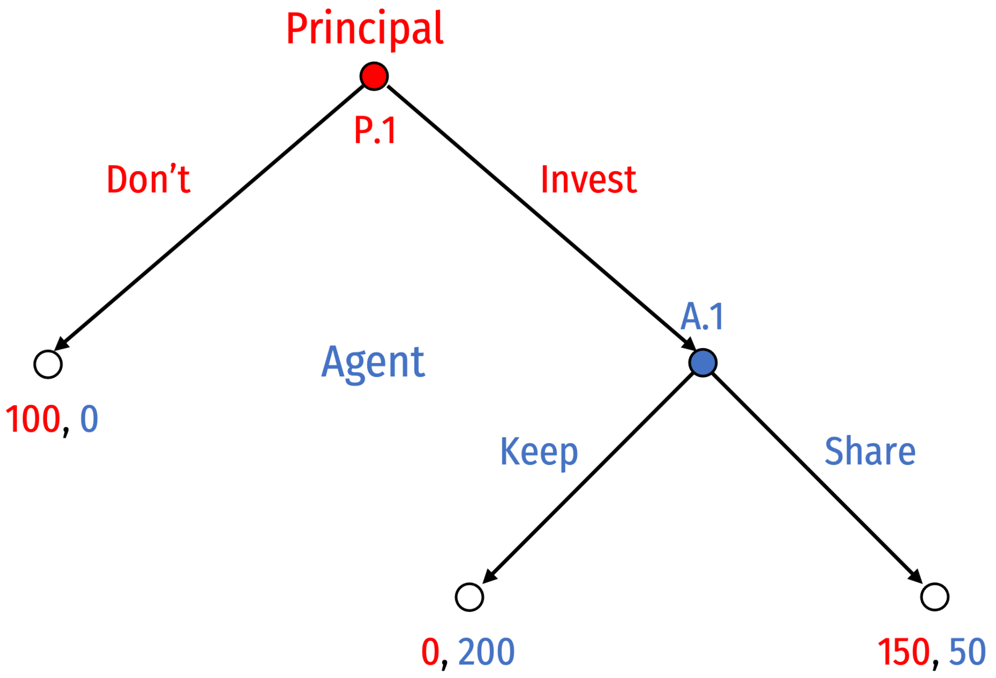
Solving the Game: Backward Induction

- We start at A.1 where **Agent** can:
  - Keep to yield outcome (0, 200)
  - Share to yield outcome (150, 50)



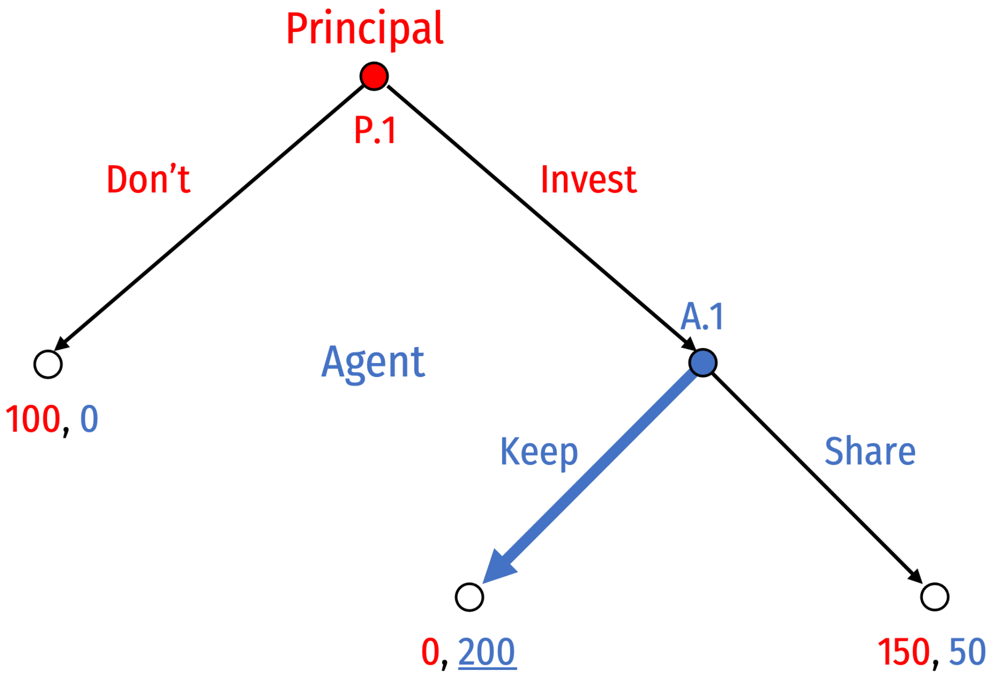
Solving the Game: Backward Induction

- We start at A.1 where **Agent** can:
  - Keep to yield outcome (0, 200)
  - Share to yield outcome (150, 50)
- **Agent** only considers their own payoff
  - (Invest, Keep) $\succ$ (Invest, Share)
  - 200 $\succ$ 50



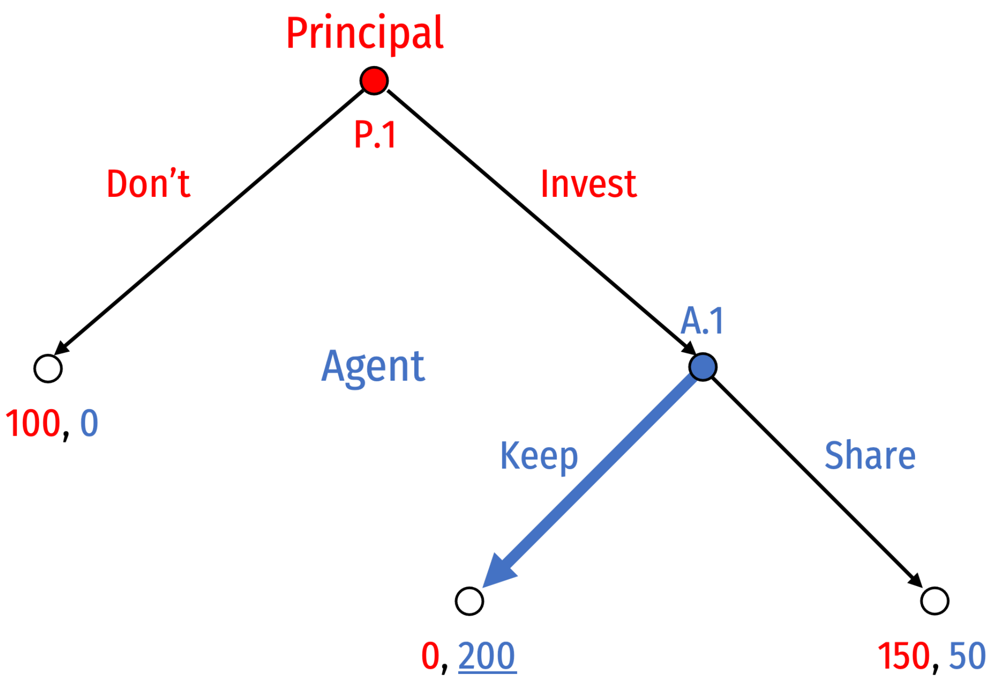
Solving the Game: Backward Induction

- **Agent** will Keep if the game reaches node A.1
- Recognizing this, what will **Principal** do?



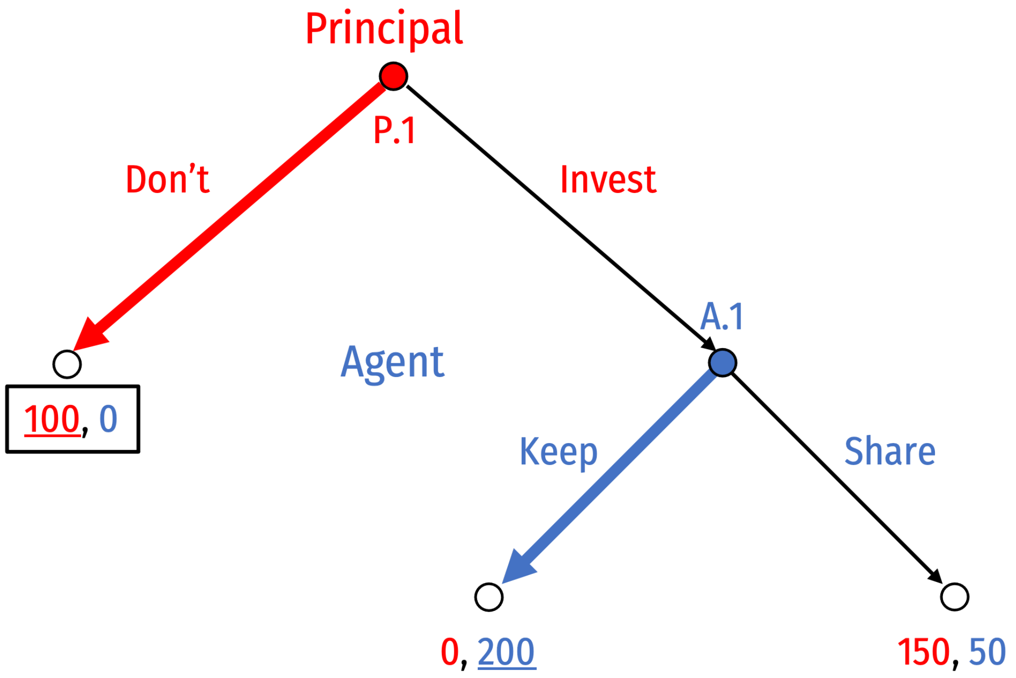
Solving the Game: Backward Induction

- Work our way up to P.1 where **Principal** can:
  - Don't to yield outcome (100, 0)
  - Invest, knowing **Agent** will Keep, to yield outcome (0, 200)
- **Principal** only considers their own payoff
  - (Don't) $\succ$ (Invest, Keep)
  - $100 \succ 0$



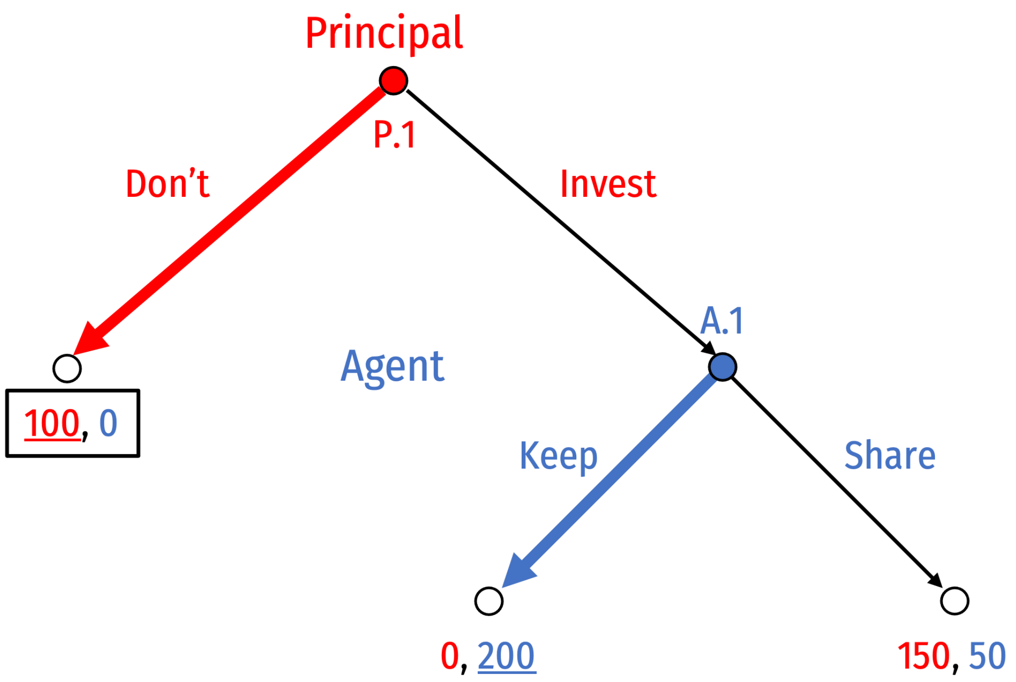
Solving the Game: Backward Induction

- **Equilibrium**: (Don't, Keep)
  - Defined by the strategy played by each player



Solving the Game: Pruning the Tree

- As we work backwards, we can **prune** the branches of the game tree
    - Highlight branches that players **will** choose
    - Cross out branches that players will **not** choose
- **Equilibrium path of play** is highlighted from the root to one terminal node
    - (Don't)
    - All other paths are not taken



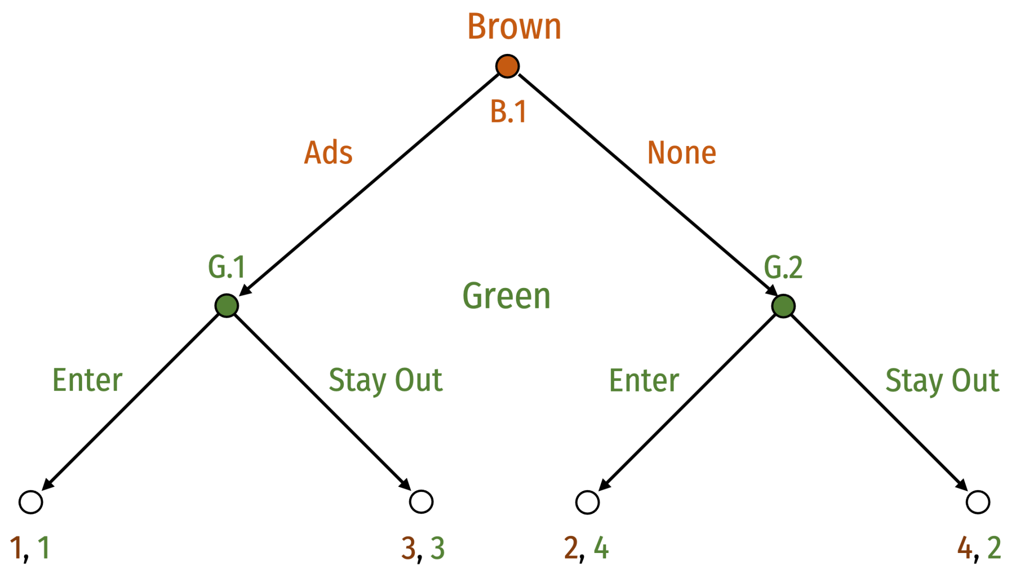
Another Example: Senate Race

- Incumbent **Senator Brown** runs for reelection

- Challenger is **Congresswoman Green**

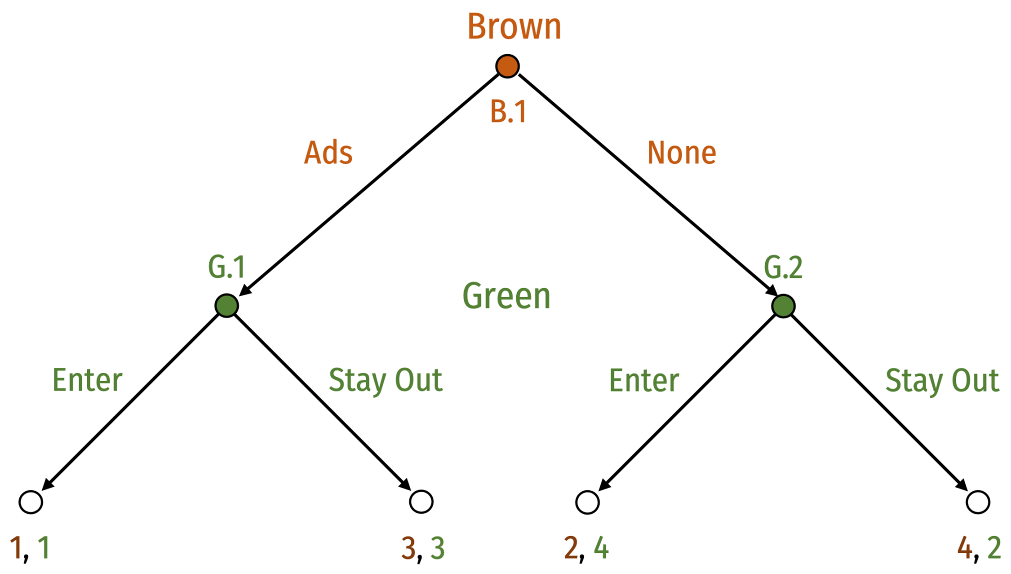
- **Brown** moves first, must decide early-on to Run Ads or No Ads

- **Green** moves second, must decide to Enter or Stay Out



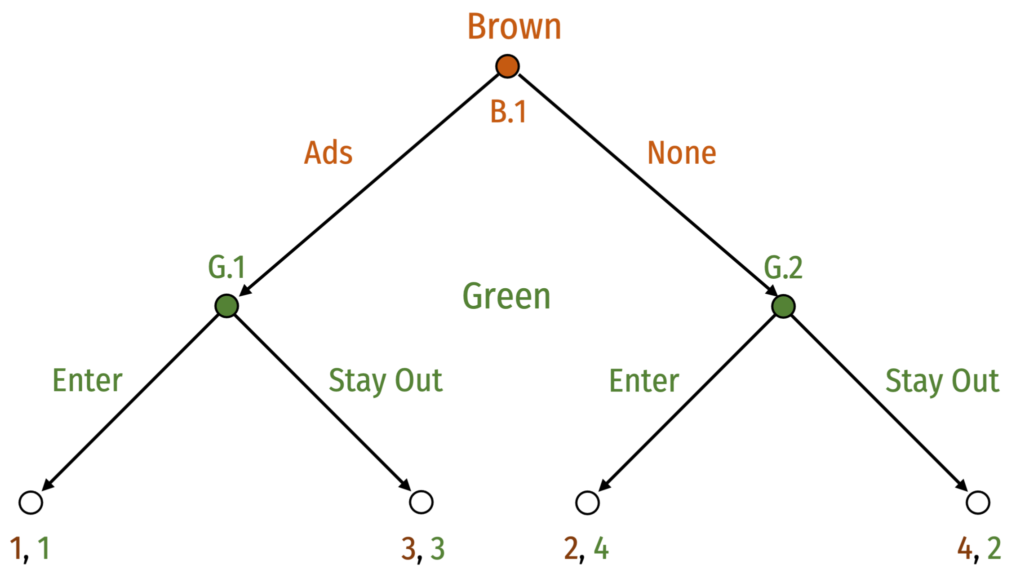
Another Example: Senate Race

- Payoff considerations:
  - Ads are costly, **Brown** would prefer to not run ads
  - **Green** will fare better if **Brown** does not run ads
- Use 1,2,3,4 for simple rankings



Senate Race Game: Strategies

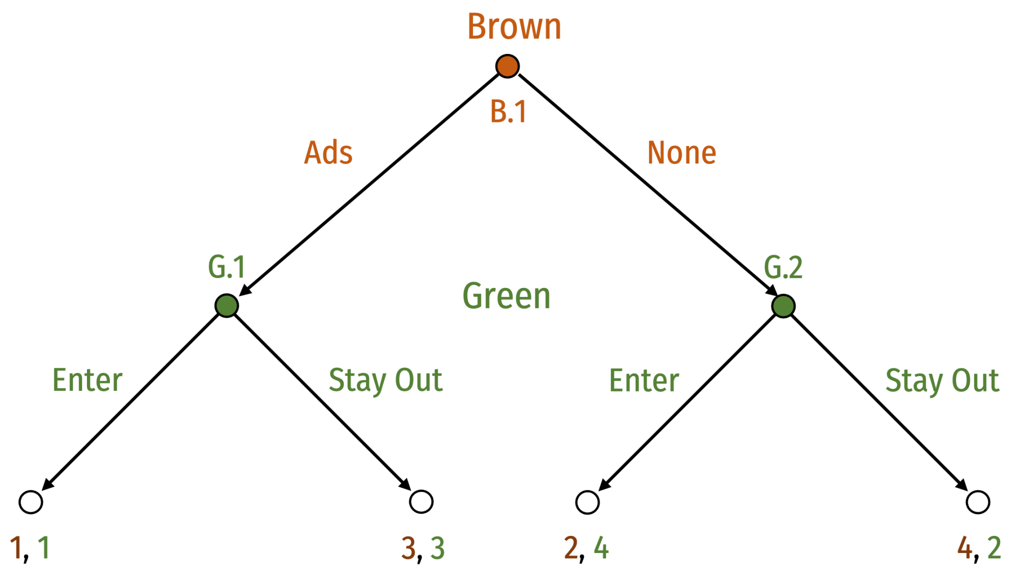
- **Brown** has 2 strategies:
    1. Ads at B.1
    2. None at B.1



Senate Race Game: Strategies

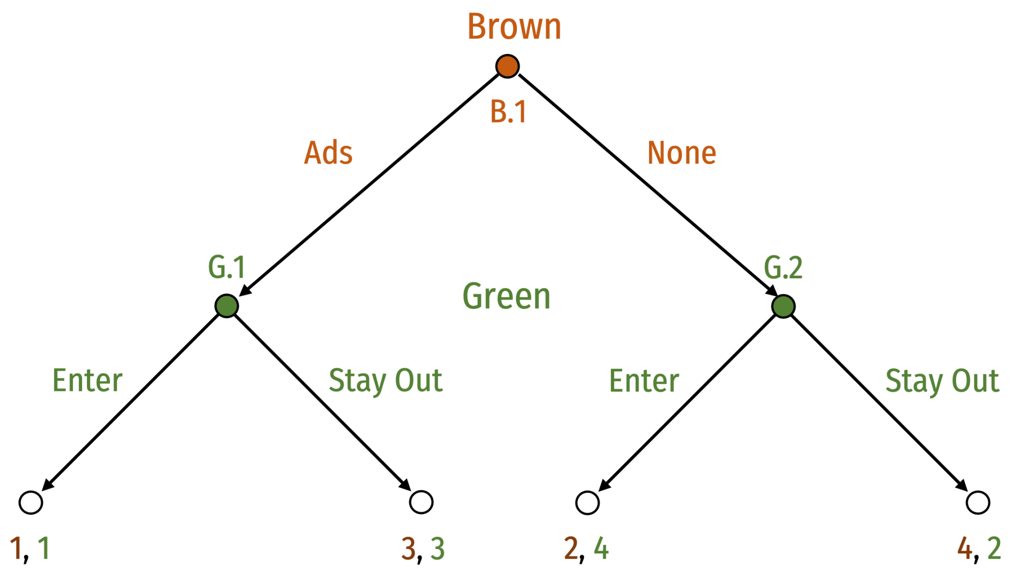
- Green has 4 strategies:
- Two decision nodes, two strategies at each node, hence $2^2 = 4$

1. Enter at G.1; Enter at G.2
2. Enter at G.1; Stay Out at G.2
3. Stay Out at G.1; Enter at G.2
4. Stay Out at G.1; Stay Out at G.2



Senate Race Game: Strategies

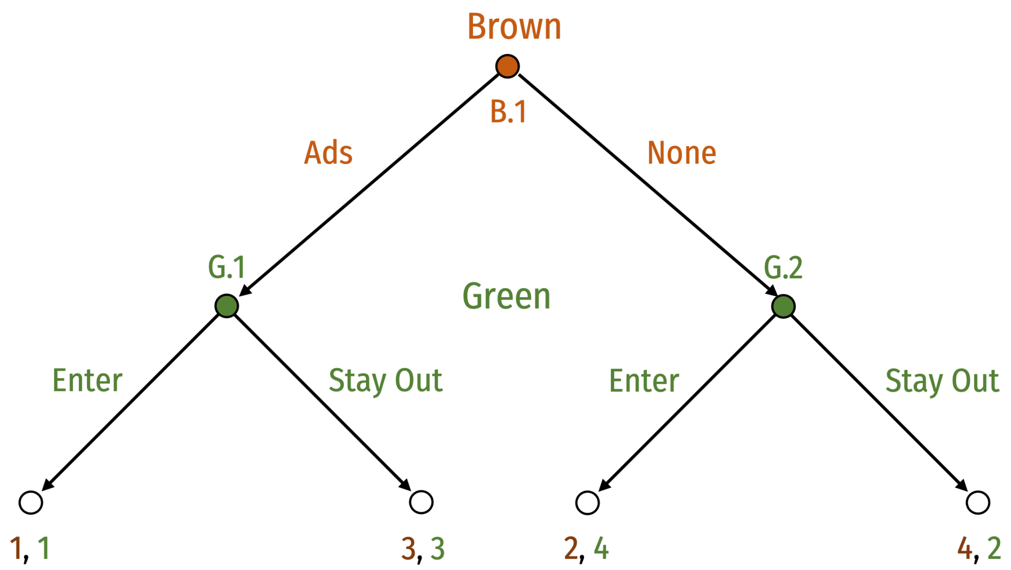
- Remember, think about a strategy like an algorithm
  1. If Ads then Enter; if None then Enter (always Enter)
  2. If Ads then Enter; if None then Stay Out
  3. If Ads then Stay Out; if None then Enter
  4. If Ads then Stay Out; if Stay Out then Stay Out (always Stay Out)



Senate Race Game: Solution

- To apply backward induction, begin with the last-mover

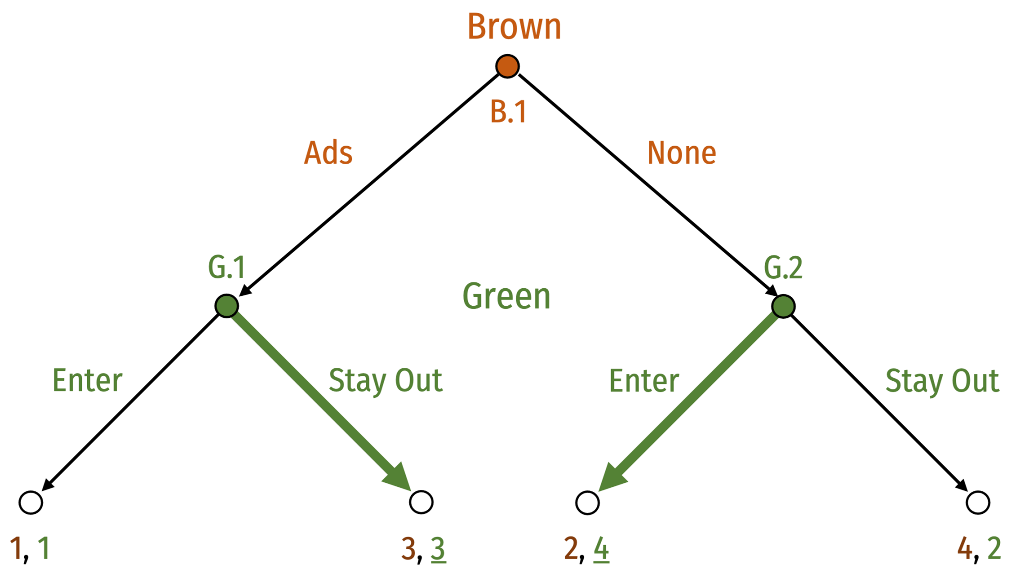
1. What will Green choose...
   - If **Brown** were to run Ads?
   - If **Brown** were to run None?



Senate Race Game: Solution

- To apply backward induction, begin with the last-mover

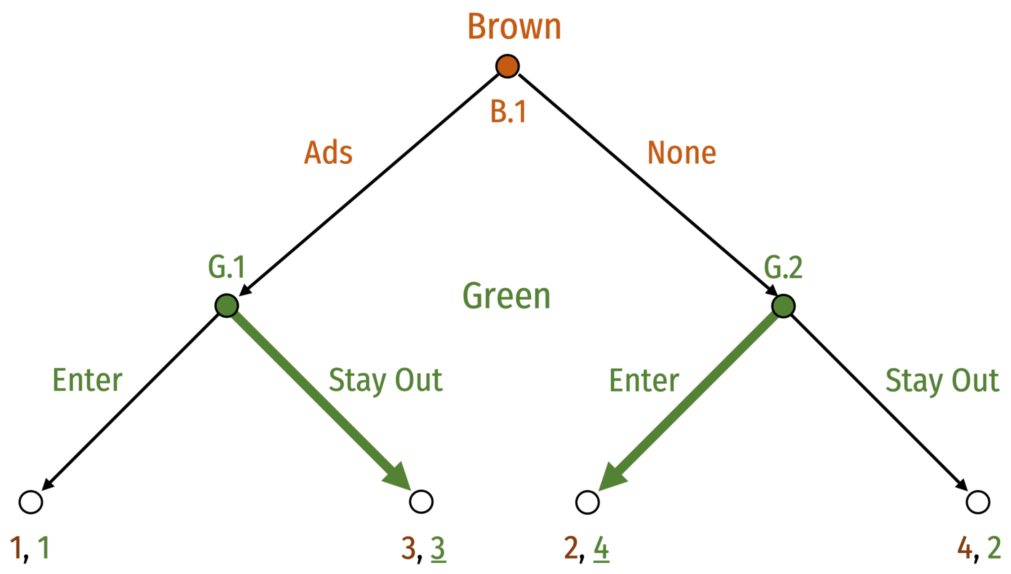
1. What will **Green** choose...
    - If Ads then Stay Out (at B.1)
    - If None then Enter (at B.2)
2. Given this, what will **Brown** choose?



Senate Race Game: Solution

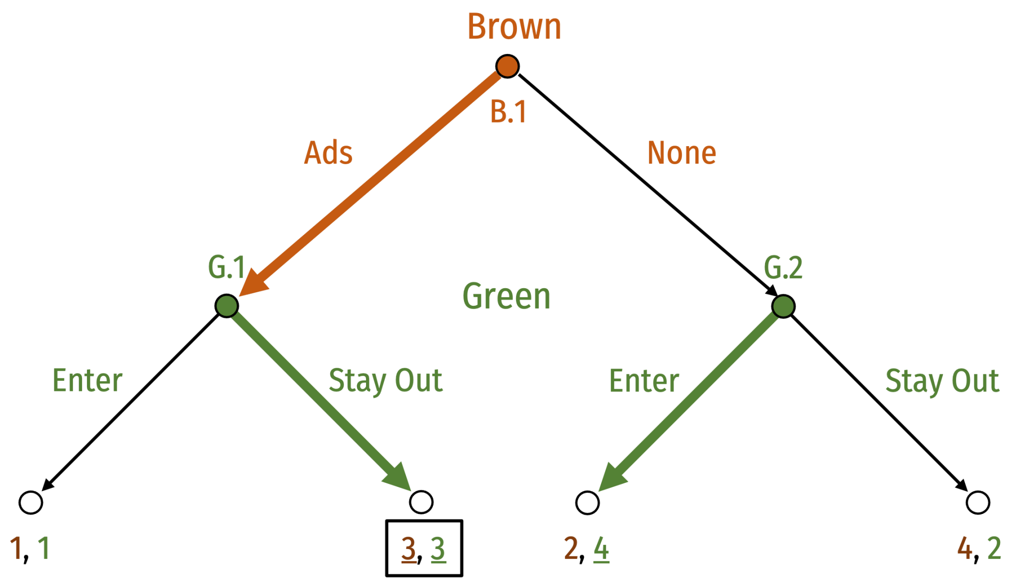
- To apply backward induction, begin with the last-mover

1. What will **Green** choose...
   - If Ads then Stay Out (at B.1)
   - If None then Enter (at B.2)
2. Given this, what will **Brown** choose?
   - (Ads, Stay Out) $\succ$ (None, Enter)
   - (3) $\succ$ (2)



Senate Race Game: Solution

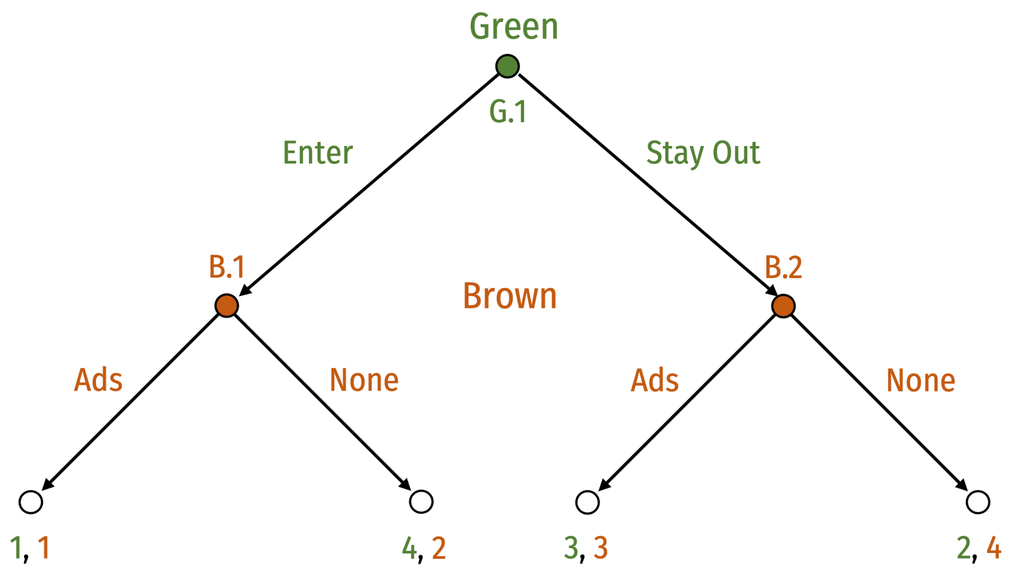
- Equilibrium: (Ads, (Stay Out, Enter))
- Notation:
  - **Brown's** strategy shows his decision at B.1 only
  - **Green**'s strategy shows her decisions at (G.1,G.2)



Mover Advantages

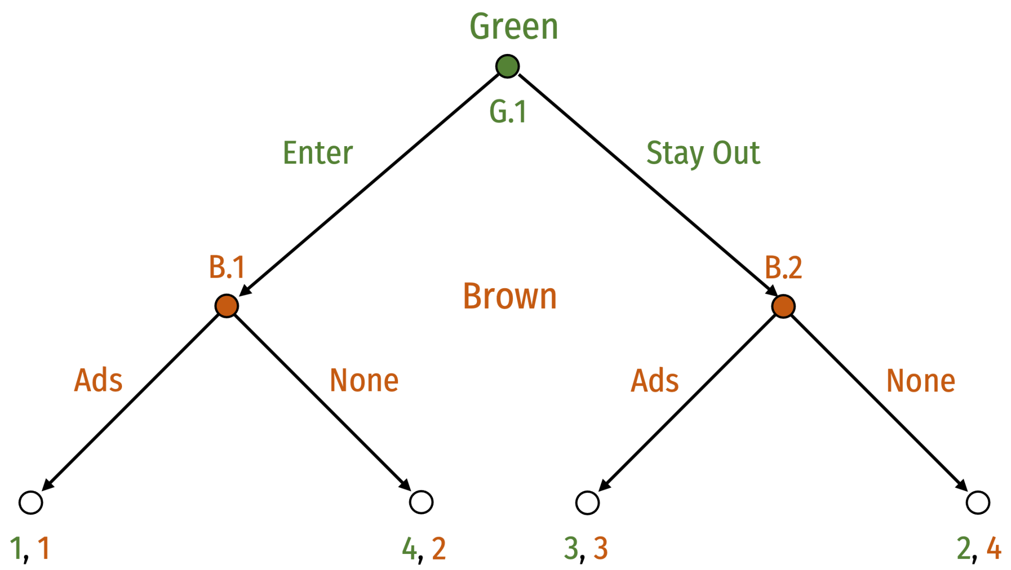
Mover Advantage: Senate Race Game

- Is there an **order advantage** to the Senate Race game?
- We saw what happens when **Brown** moves first
- Change the rules so that **Green** moves first and see what changes
    - Be careful how you write the payoffs!



Mover Advantage: Senate Race Game

- Green has 2 strategies:
    1. Enter at G.1
    2. Stay Out at G.1



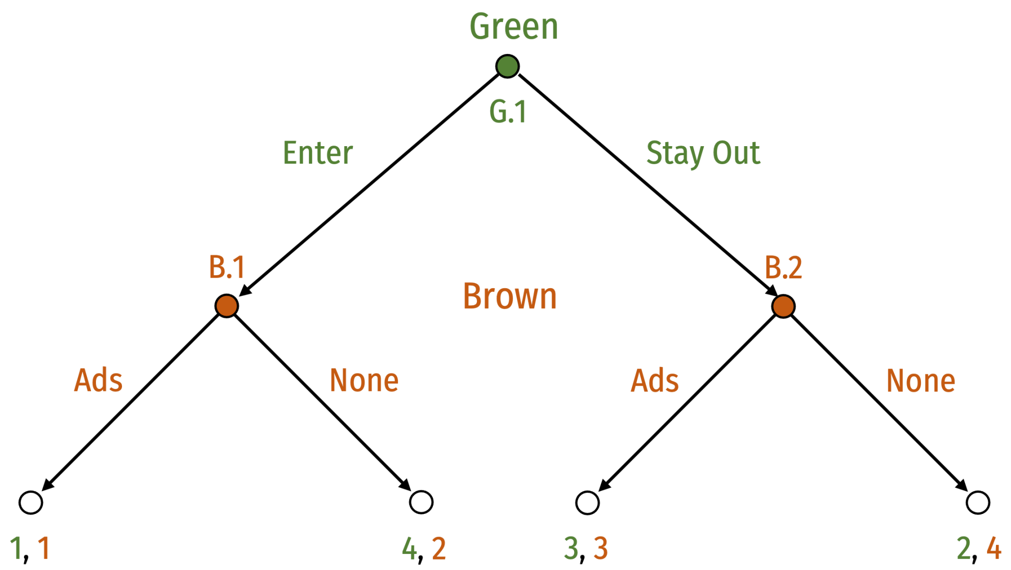
Mover Advantage: Senate Race Game

- Green has 2 strategies:

  1. Enter at G.1
  2. Stay Out at G.1

- Brown has 4 strategies:

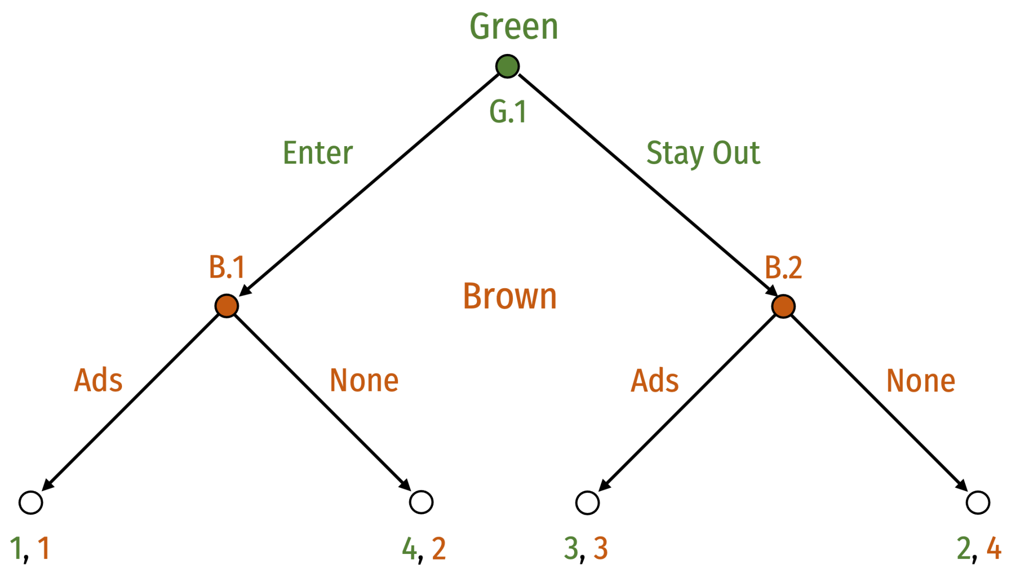
  1. Ads at B.1; Ads at B.2
  2. Ads at B.1; None at B.2
  3. None at B.1; Ads at B.2
  4. None at B.1; None at B.2



Mover Advantage: Senate Race Game

- Apply backwards induction

1. What will **Brown** choose
   - If **Green** were to Enter?
   - If **Green** were to Stay Out?



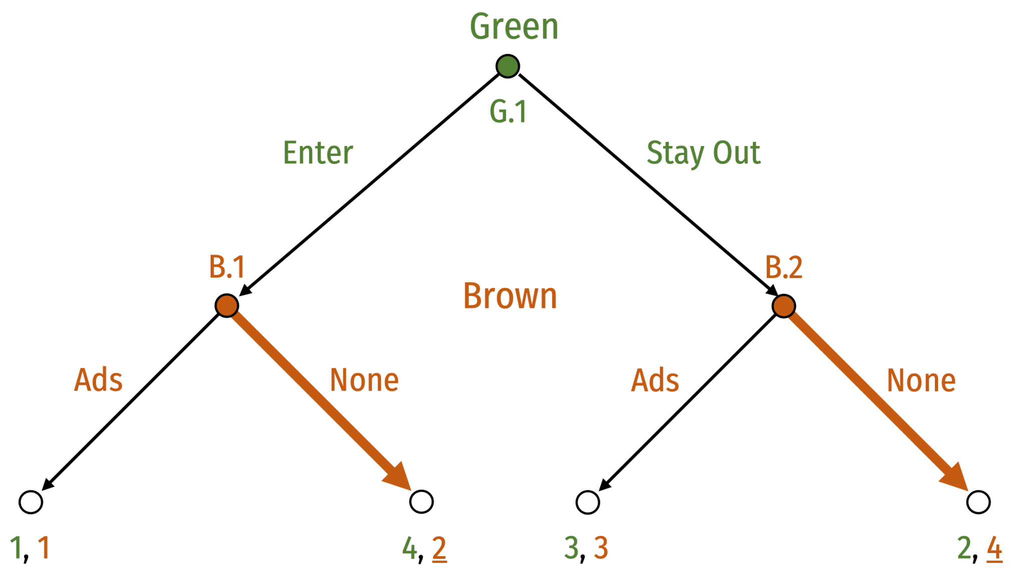
Mover Advantage: Senate Race Game

- Apply backwards induction

1. What will **Brown** choose
    - If Enter then None
    - If Stay Out then None

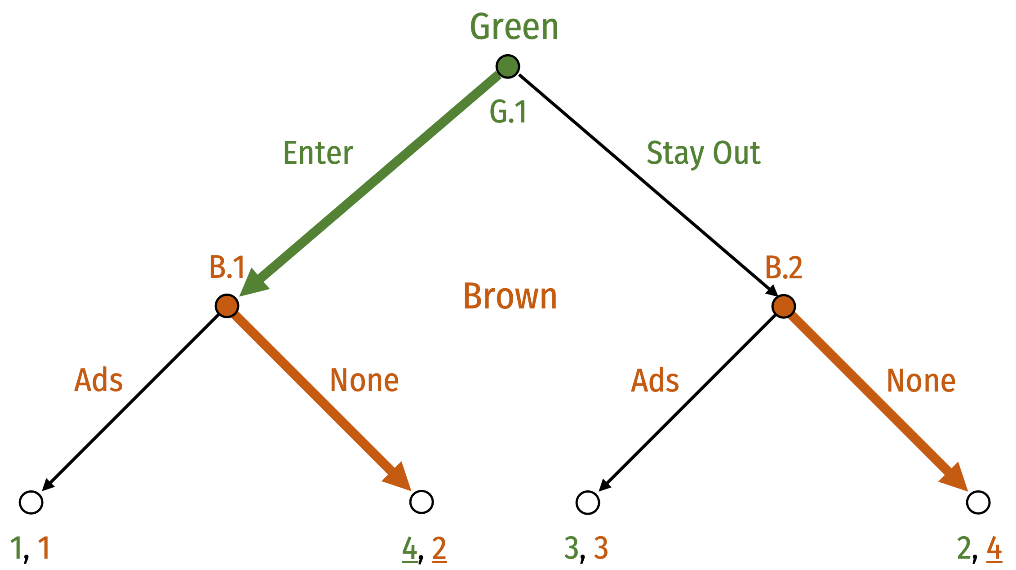
- Note None becomes a **dominant strategy** for **Brown**!

1. Given this, what will Green choose?



Mover Advantage: Senate Race Game

- Equilibrium: (Enter, (None, None))
    - Payoffs of (4, 2)
- Recall original outcome (Ads, (Stay Out, Enter))
    - Payoffs of (3, 3)
- **Brown** is worse-off moving second vs. first; **Green** is better off moving first vs. second



When Order Matters

- In general, to see if order matters, reverse sequence of moves and see if outcomes differ

- Games with first-mover advantage:
    - Century Mark
    - Tic-tac-toe
    - Chess? Checkers?

- Games with second-mover advantage:
    - Free riders
    - Business?

When Order Matters

Clayton Christensen

> “When you look across the sweep of business history, most companies that once seemed successful—the best practitioners of best practice—were in the middle of the pack (or, worse, the back of it) a decade or two later...What often causes this lagging behind are two principles of good management taught in business schools: that you should always listen to and respond to the needs of your best customers, and that you should focus investments on those innovations that promise the highest returns. But these two principles, in practice, actually sow the seeds of every successful company's ultimate demise,” (ix-x).

Christensen, Clayton, 2016[1997], *The Innovator's Dilemma: When New Technologies Cause Great Firms to Fail*

When Order Matters

Peter Thiel

> “You've probably heard about 'first mover advantage': if you're the first entrant into a market, you can capture significant market share while competitors scramble to get started. But moving first is a tactic, not a goal...[B]eing the first mover doesn't do you any good if someone comes along and unseats you. It's much better to be the *last* mover—that is, to make the last great development in a specific market and enjoy years or even decades of monopoly profits.,” (57-58).

Thiel, Peter, 2014, *Zero to One: Notes on Startups or How to Build the Future*

Adding Players

Adding Players

Adding Players

Adding Players

Adding Players

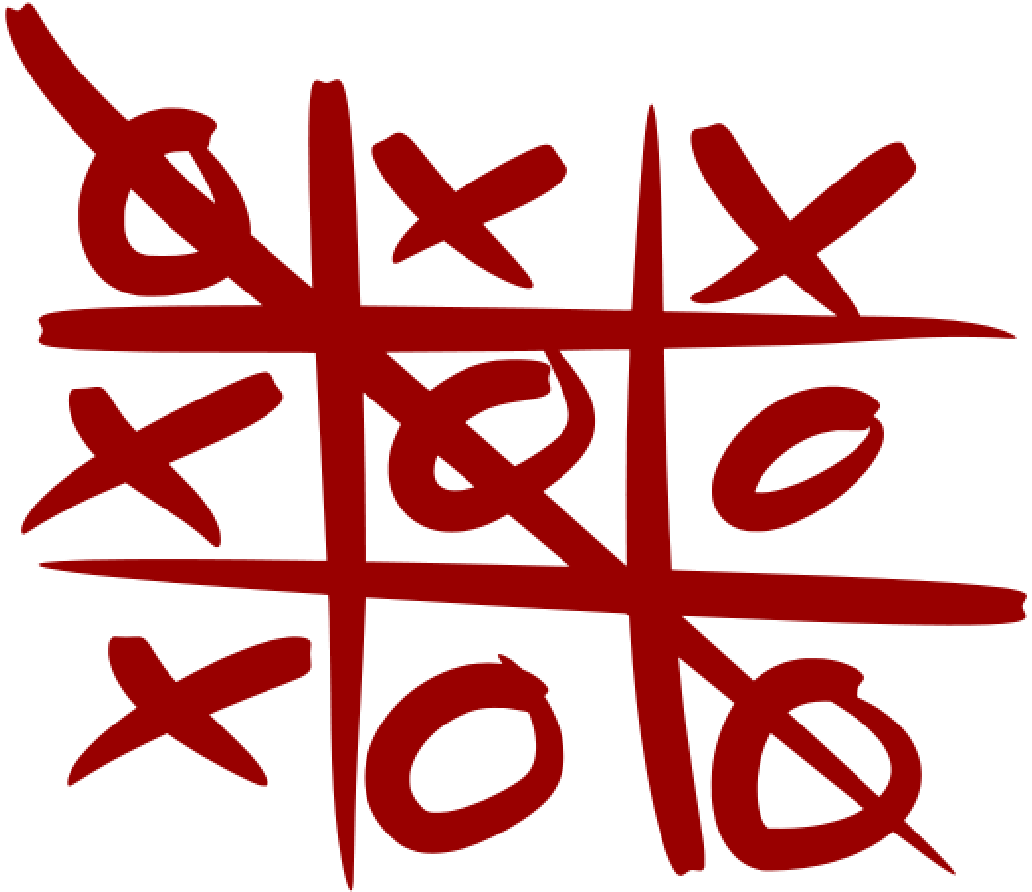
Summary

1. Construct a game tree
    - Place players in proper order
    - Specify which decisions are available to each player at each decision node
    - Specify payoffs to all players in terminal noides
2. Solve for rollback equilibrium
    - Start with last-mover, identify best response, prune all other branches
    - Work successively backwards to the root
    - Highlight equilibrium path of play

How Reasonable is Rollback Thinking?

How Reasonable is Rollback Thinking?

- Useful for simple games with few players & moves
- More difficult for complex games (more moves and/or players)
  - Tic-tac-toe has $9!$ or $362,880$ possible moves



How Reasonable is Rollback Thinking?

- Chess estimated to have 10^{120} possible moves
- Players need rules to assign "payoffs" to *non-terminal* nodes, an "intermediate value function"
  - Humans $<$ computers at anticipating future moves
  - Humans $>$ computers at mid-game intuition and experience

Garry Kasparov vs. IBM's Deep Blue

How Reasonable is Rollback Thinking?

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