4.4 — Incomplete & Asymmetric Info.

ECON 316 • Game Theory • Fall 2021

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Outline

Information in Games

Information

- **Perfect information**: all players know the rules and all possible strategies, payoffs, and move history of other players

- **Common knowledge** assumption: Player 1 knows that Player 2 knows that Player 1 knows that ...

Information

- Imperfect information: players all know the game, but don't know what other players are choosing
  - “Strategic uncertainty”
  - Seen as simultaneous games
- Incomplete information: players don't have all the information about the game
  - “External uncertainty”
  - Who the other players are, what payoffs are, etc.

Simultaneous Games and Imperfect Information

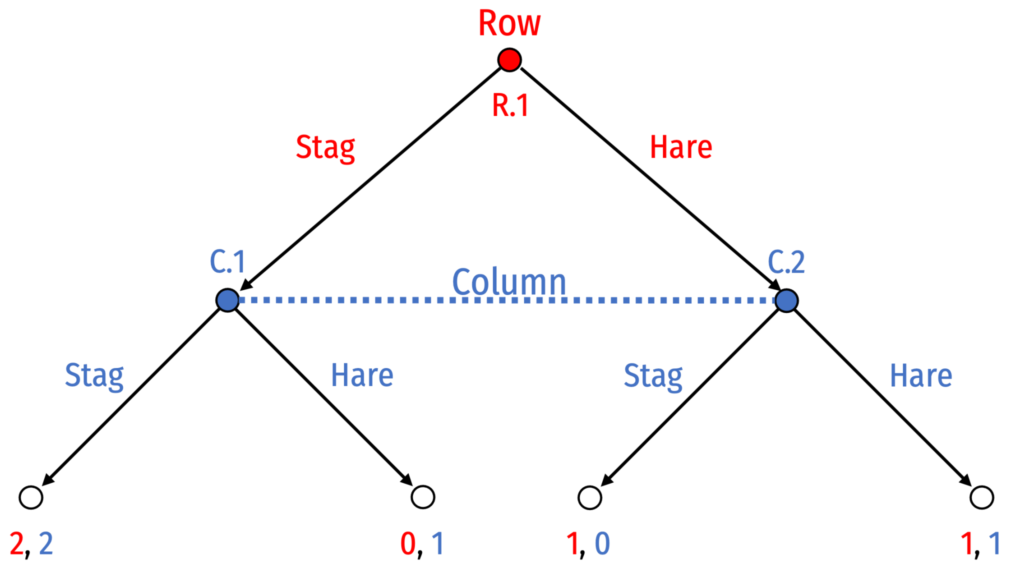
- Let’s consider the simultaneous-move **Stag Hunt** in **strategic form**

- We can't model this as an *extensive form* game with perfect information
  - Each player doesn’t know what the other chose

		Column	
		Stag	**Hare**
Row	**Stag**	2 2	0 1
	Hare	1 0	1 1

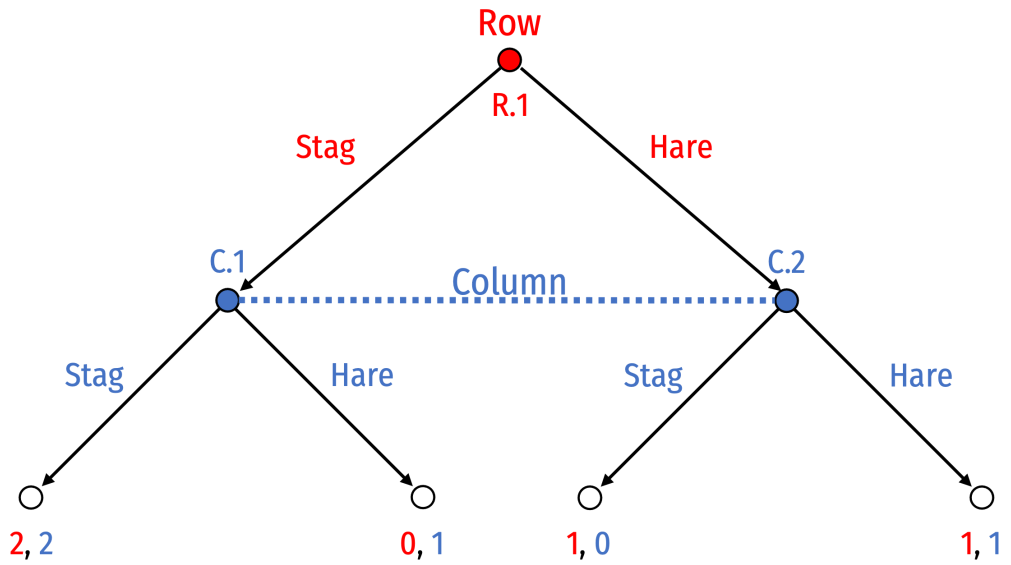
Simultaneous Games and Imperfect Information

- We *can* model it in **extensive form** with **imperfect information**
  - Let Row move first (order really doesn't matter with symmetric payoffs)
- Row's move is hidden from Column:
  - Can't distinguish between the **history** where Row chose Stag and the history where Row chose Hare.



Simultaneous Games and Info Sets

- **Information set** (dotted line/oval connecting decision nodes) for Column $\implies$ can't distinguish between histories of Stag or Hare
- Column doesn't know if they are deciding at node C.1 or C.2 (whether Row has played Stag or Hare)



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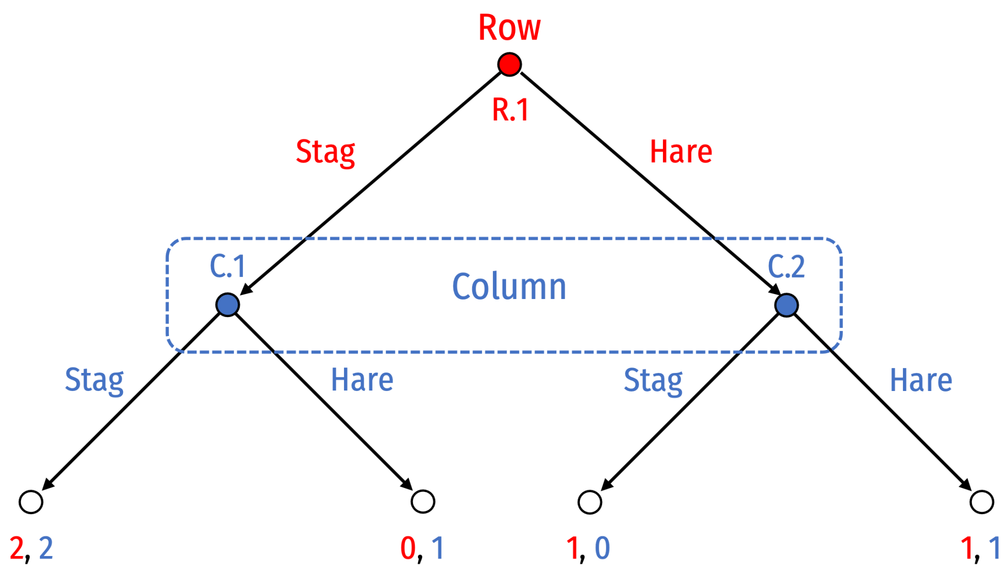
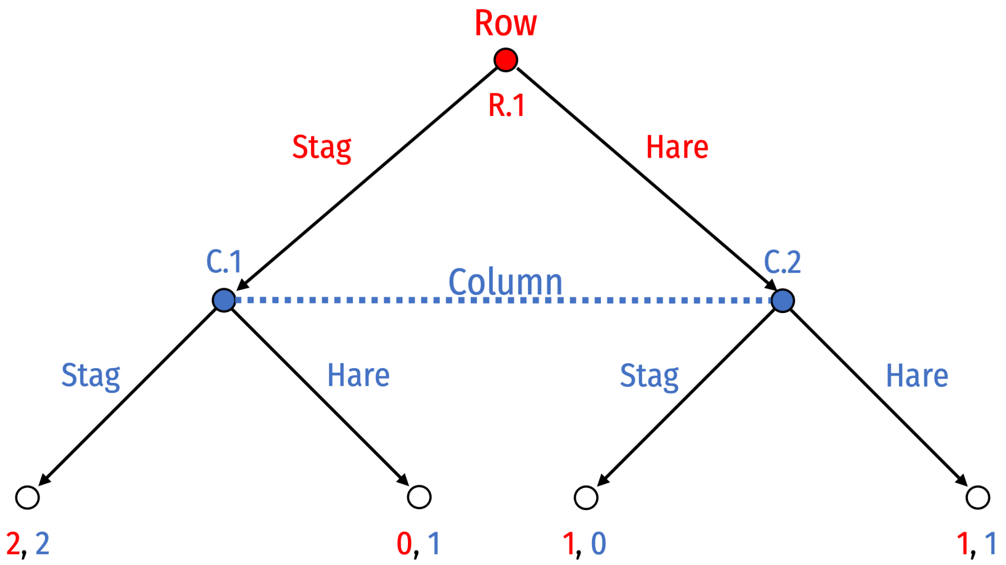
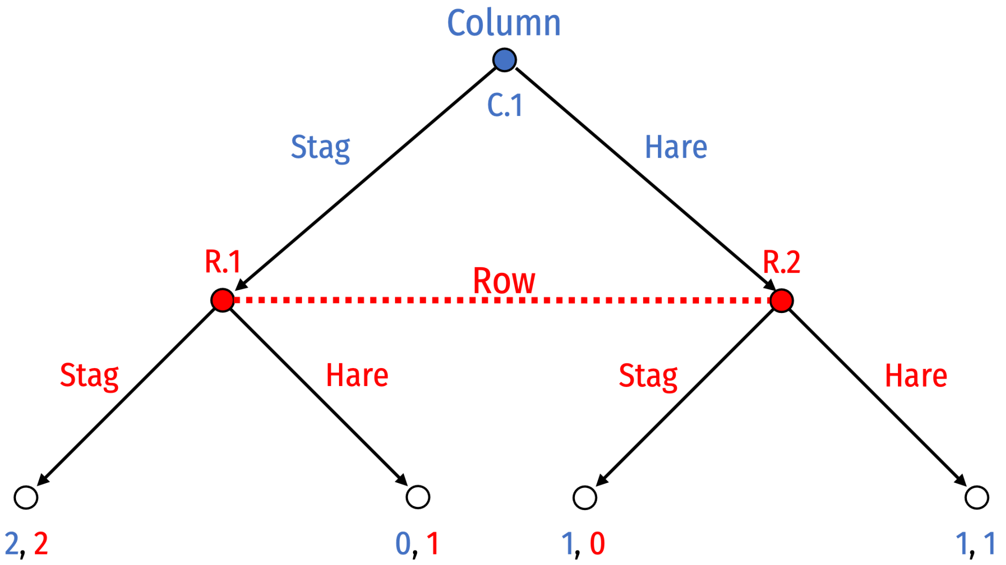


Simultaneous Games: No Mover Advantage



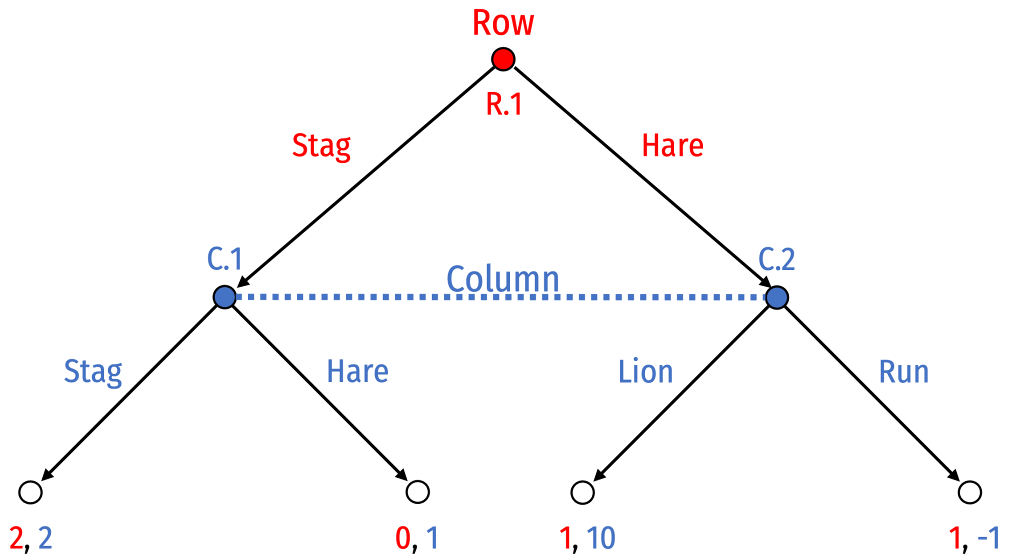


- Note changing who moves first here makes no difference on the game, since "second-mover" still does not know what "first-mover" chooses!

Strategies and Information Sets

- Strategies available to player within an information set must be the same across all decision nodes/histories

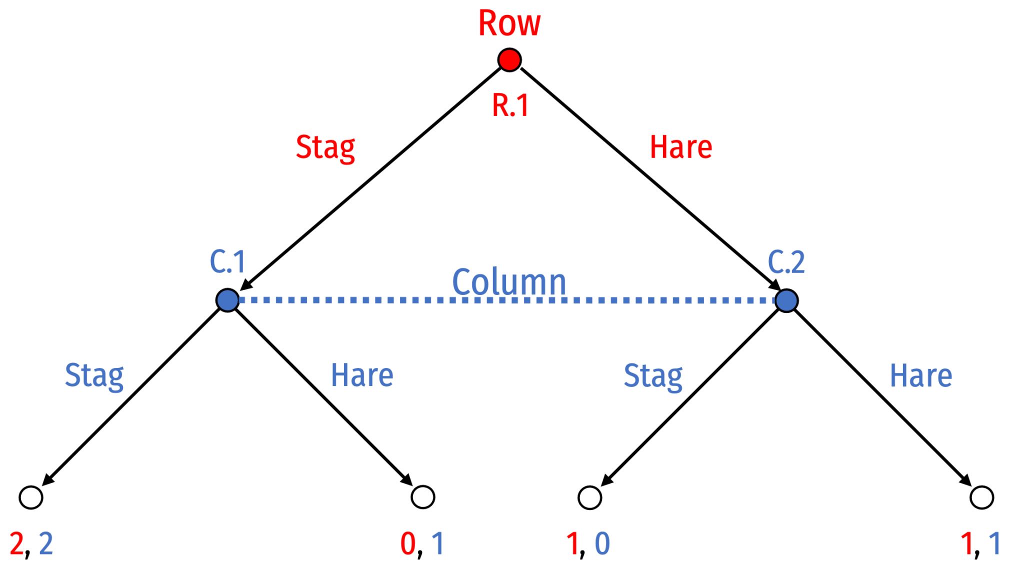
- If they are different, player can tell which history they are on given the unique strategies available to them
    - Column would clearly know if they are at node C.1 or C.2 since strategies available are different!



This is not a valid game

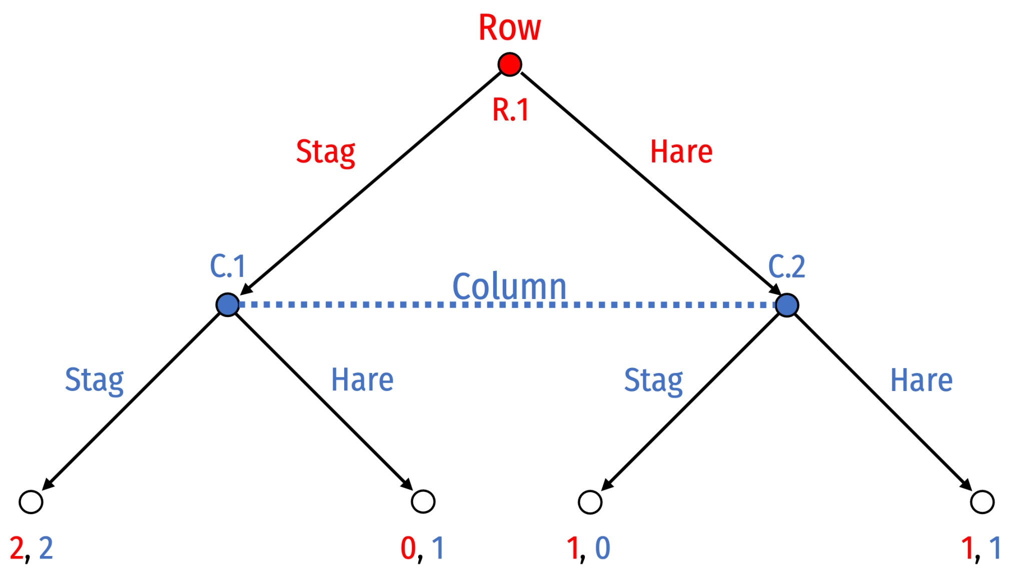
Strategies and Information Sets

- Furthermore, Column must play the *same* strategy across the decision nodes
    - (i.e. *always* Stag or *always* Hunt) at *both* (C.1 and C.2)
    - Can't play (Stag, Hare) or (Hare, Stag) at (C.1,C.2)
- Again, doesn't know what decision node they are actually deciding at



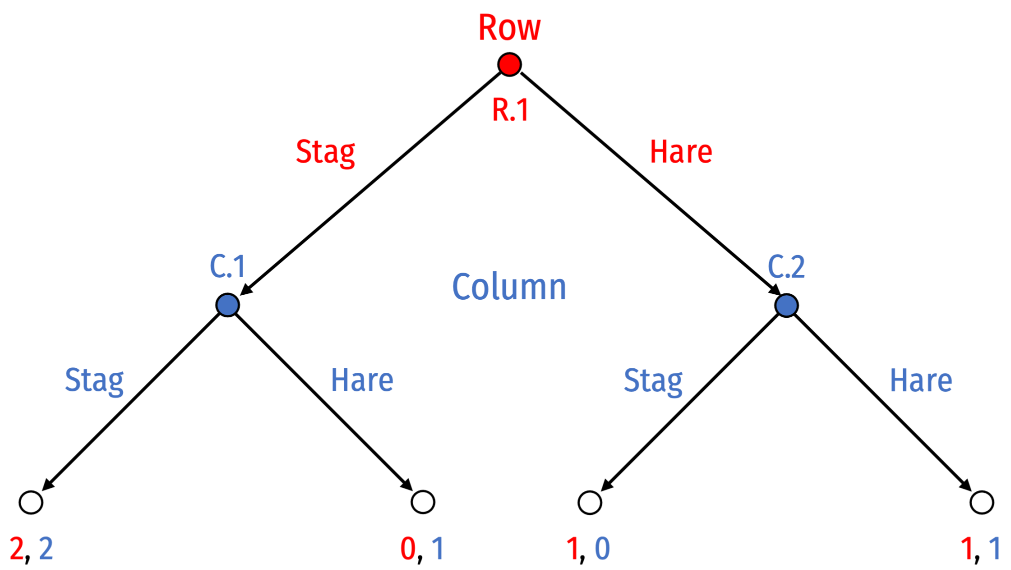
Strategies and Information Sets

- Clarify what we mean by **strategy**: a complete plan of action of all the decisions a player will make at every possible **information set**
  - (rather than merely every decision node)
- Until now, information sets have consisted of a single decision node ("singleton")



Perfect Information

- We can now more precisely define **perfect information**: no information sets contain multiple decision nodes (are all "singleton" nodes)
- Individual can differentiate between histories of game at each decision node

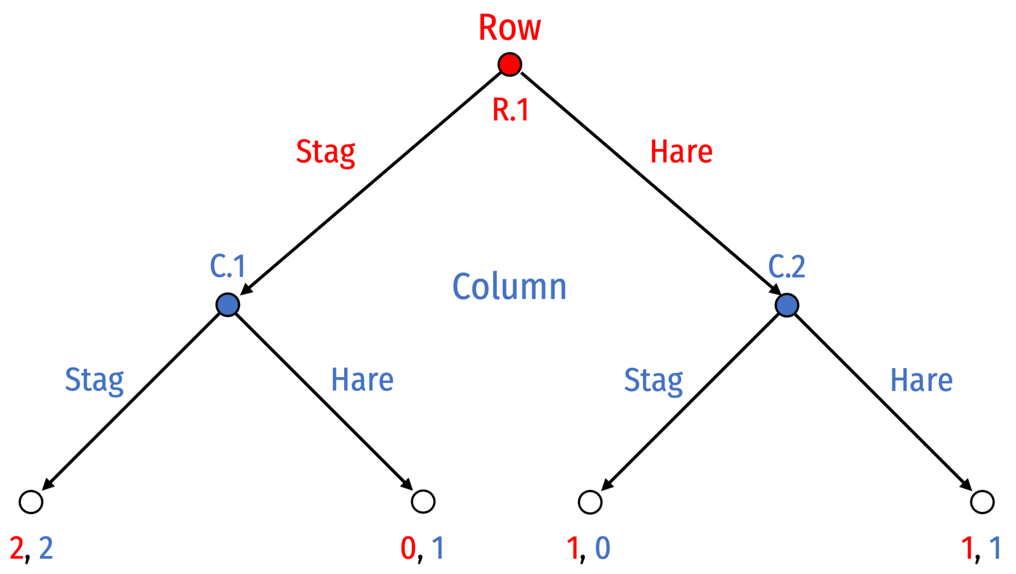


Stag Hunt with Perfect Information

- With **perfect information**, Column's strategies can be conditional on what Row plays

  1. (Stag, Stag)
  2. (Stag, Hare)
  3. (Hare, Stag)
  4. (Hare, Hare)

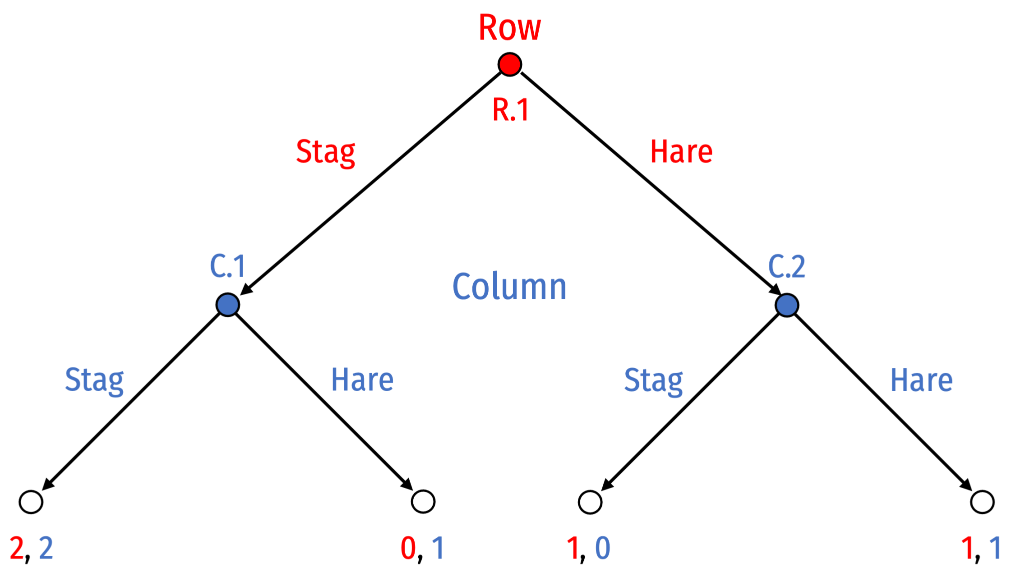
- Each information set is a singleton (i.e. nodes C.1 and C.2 each contain a separate information set)



Stag Hunt with Perfect Information

- Using normal form, three Nash equilibria with **perfect information**:

1. {Stag, (Stag, Stag)}
2. {Stag, (Stag, Hare)}
3. {Hare, (Hare, Hare)}

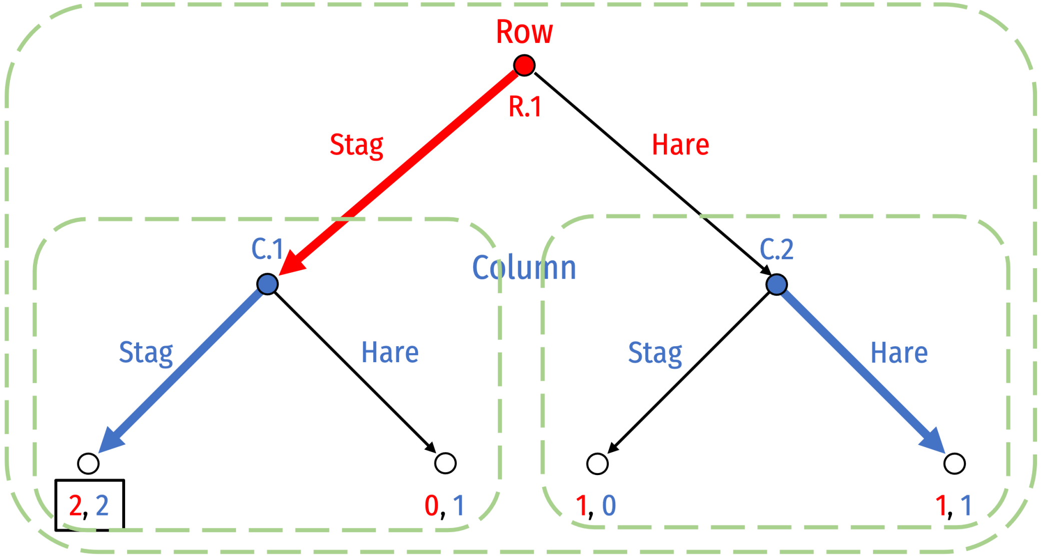


Stag Hunt with Perfect Information

- Using normal form, three Nash equilibria with **perfect information**:

1. {Stag, (Stag, Stag)}
2. **{Stag, (Stag, Hare)}**
3. {Hare, (Hare, Hare)}

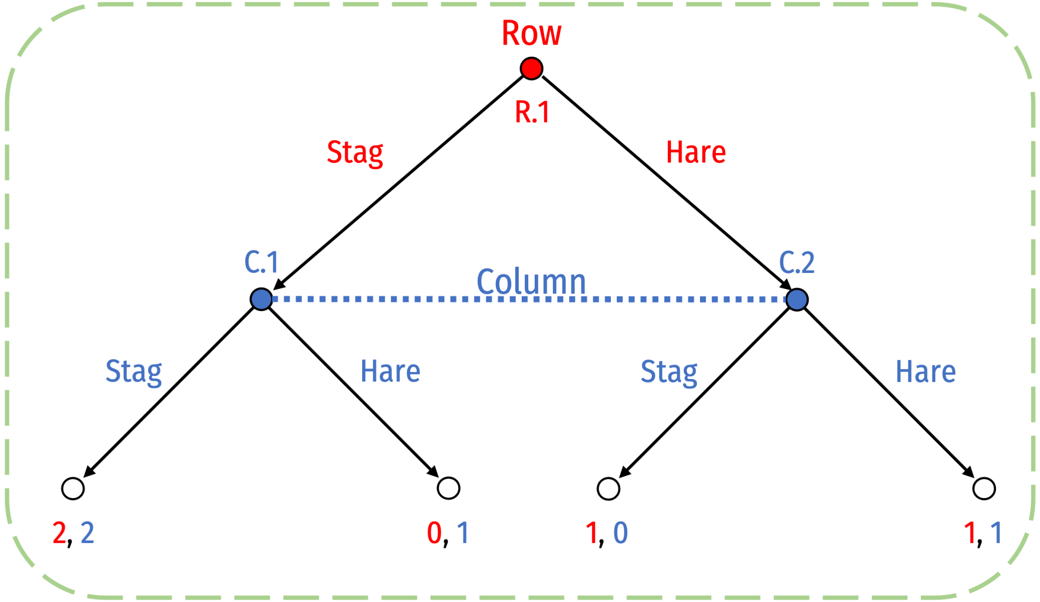
- Only #2 {Stag, (Stag, Hare)} is **subgame perfect**



		Column			
		(S, S)	(S, H)	(H, S)	(H, H)
Row	Stag	2, 2	2, 2	0, 1	0, 1
	Hare	1, 0	1, 1	1, 0	1, 1

Imperfect Information and Subgame Perfection

- With **imperfect information**, some information sets contain multiple decision nodes
- A subgame must contain all nodes in the information set, cannot "break" information sets
  - Nodes 2.1 and 2.2 do *not* initiate subgames (breaks information sets)
  - The *only* subgame possible is the overall game itself (contains all nodes in information set)
- We **cannot** use subgame perfection as a solution concept here



Imperfect Info. May Imply A Simultaneous Game

- Column cannot play any *conditional* strategies depending on what Row does
  - Can't know what Row will play!
  - Must choose an *unconditional* strategy to always play Stag or Hare
- All we can do is solve the game via strategic form as usual

		Column: Stag	Column: Hare
Row	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

Incomplete Information

Incomplete Information

- Even in games with imperfect information (e.g. simultaneous games), we have assumed information was **complete**
    - Players know the rules, strategies available, and the payoffs to each player
- Source of uncertainty was **strategic**: players didn't know the **history** of the game (moves made by other players)

Incomplete Information

- Now consider games with **incomplete information**
  - Players don't know something about the game
  - Common example: what the payoffs to the other player are (but know your own)
- Textbook calls this **external uncertainty**: the game is not fully clear due to some undetermined external factors

External Uncertainty: Playing with Nature

- We can deal with external uncertainty by including **Nature** as a player
- **Nature** has no strategic interest in the outcomes (has no payoff and no objectives)
- Really just a metaphor for rolling (possibly weighted) dice

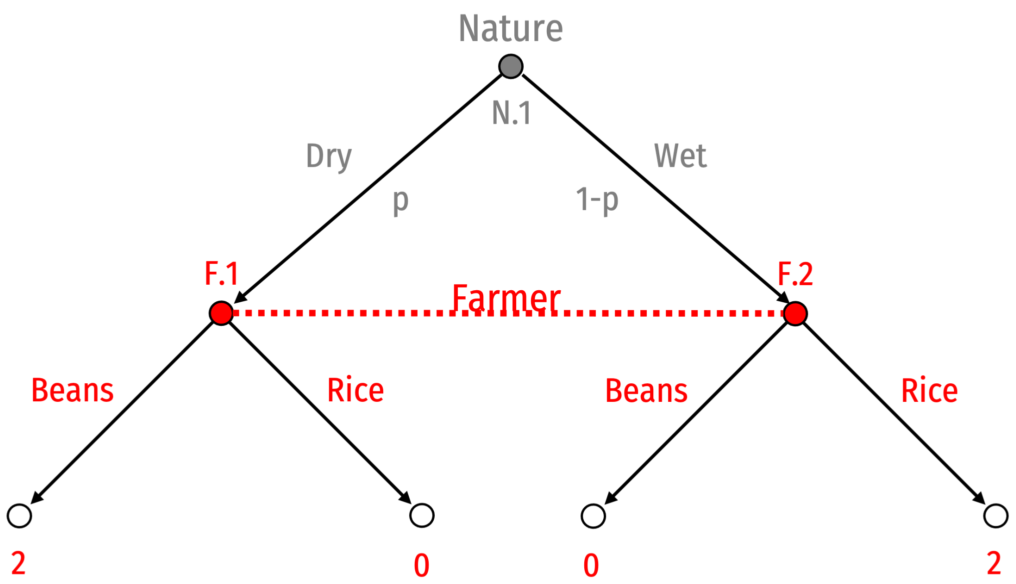
External Uncertainty: Simple Example

- Consider a Farmer who (ignoring competition) must determine what crops to plant: Beans, which do better in dry seasons; or Rice which does better in wet seasons

- Let **Nature** decide what the weather this season will be

    - With some probability p, **Nature** will "choose" a **wet** season

- Farmer can't know what **Nature** chose



External Uncertainty: Simple Example

- Farmer must maximize **expected payoff**
- Consider a mixed strategy "against" **Nature**

		Nature: Dry (p)	Nature: Wet (1-p)	
Farmer	Beans	2	0	2p
	Rice	0	2	2-2p

External Uncertainty: Simple Example

- Farmer must maximize **expected payoff**
- Consider a mixed strategy "against" **Nature**

$$
\begin{aligned}
E[Beans] &= E[Rice] \\
2p &= 2 - 2p \\
p^\star &= 0.50
\end{aligned}
$$

- If $p > 0.50$, Farmer should plant Rice
- If $p < 0.50$, Farmer should plant Beans

		Nature: Dry (p)	Nature: Wet $(1-p)$	
Farmer	Beans	2	0	2p
	Rice	0	2	2-2p

External Uncertainty: Simple Example

- Now suppose Farmer can estimate (based on experience, forecasts, etc) p to be 0.40
- Now Farmer has a pure strategy "against" **Nature**

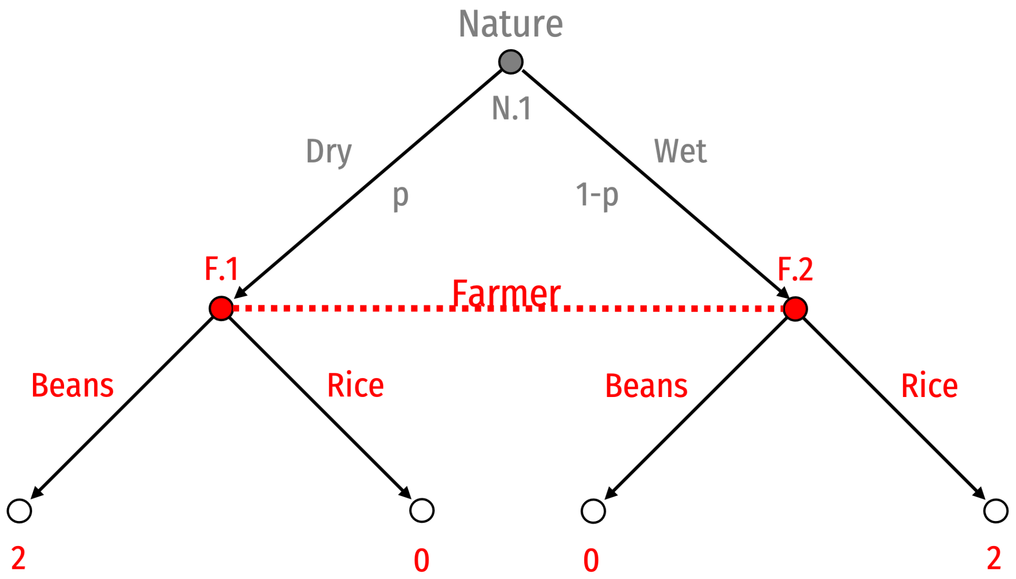
$$E[Beans] < E[Rice]$$
$$(0.40)2 + (0.60)0+ \; < (0.40)0 + (0.60)2$$
$$0.80 < 1.20$$

- Definitely plant Rice

		Nature: Dry (0.40)	Nature: Wet (0.60)	
Farmer	**Beans**	2	0	0.80
	Rice	0	2	1.20

External Uncertainty: Simple Example

- Key here is Farmer's **beliefs** about p

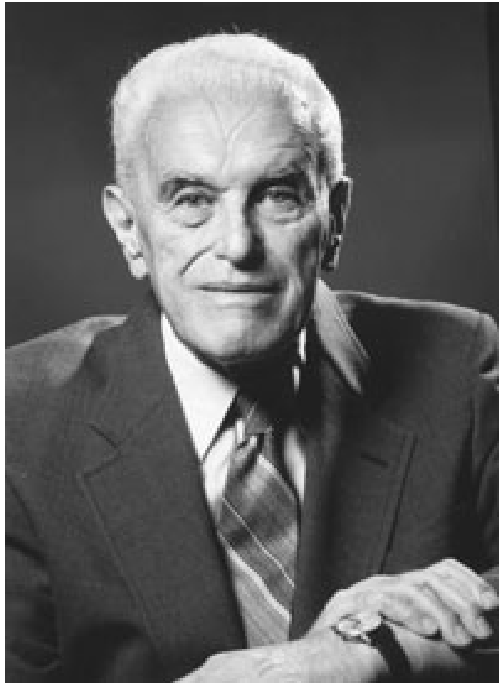


Asymmetric Information & Simultaneous Bayesian Games

Asymmetric Information

- A particular type of incomplete information is **asymmetric information**, where players might not know all relevant information about others
  - Other player's strategies, payoffs, preferences, or "type"
- Typically, one player has important **private information** about themself that other players are not privy to
  - e.g. Player 2 knows their "type" but Player 1 does not know Player 2's type

Asymmetric Information



John C. Harsanyi

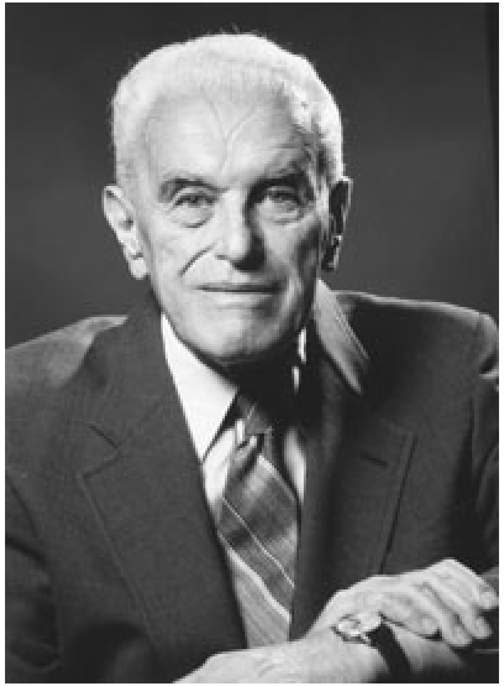
1920—2000

Economics Nobel 1994

> "[T]he original game can be replaced by a game where nature first conducts a lottery in accordance with the basic probablity distribution, and the outcome of this lottery will decide which particular subgame will be played, i.e., what the actual values of the relevant parameters will be in the game. Yet each player will receive only partial information about the outcome of the lottery, and about the values of these parameters," (p.159).

Harsanyi, John C, 1976, "Games with Incomplete Information Played by 'Bayesian' Players I-III, Part I: The Basic Model," *Management Science* 14(3): 159—182

Asymmetric Information



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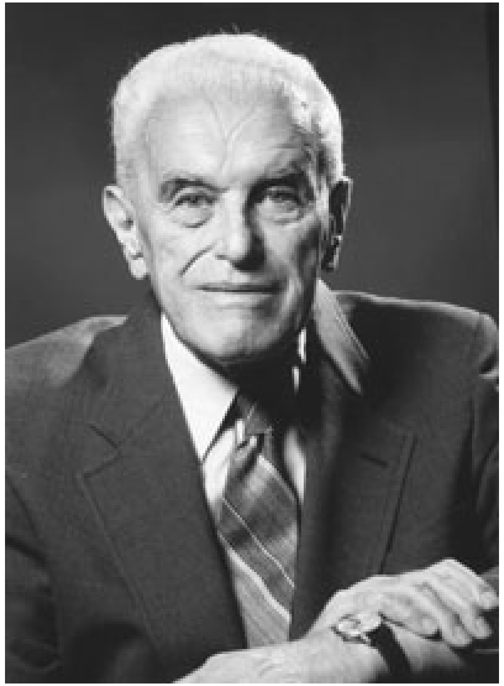
> “In such a game player 1's strategy choice will depend on what he expects (or believes) to be player 2's payoff function U_2, as the nature of the latter will be an important detemunant of player 2's behavior in the game...If we follow the Bayesian approach and represent the players' expectations or beliefs by subjective probability distributions, then player 1's first-order expectation will have the nature of a subjective probability distribution $P_1^1(U_2)$ over all alternative payoff functions U_2 that player 2 may possibly have. Likewise, player 2's first-order expectation will be a subjective probability distribution $P_2^1(U_1)$ over all alternative payoff functions U_1 that player 1 may possibly have,” (pp.163—164).

Asymmetric Information



John C. Harsanyi

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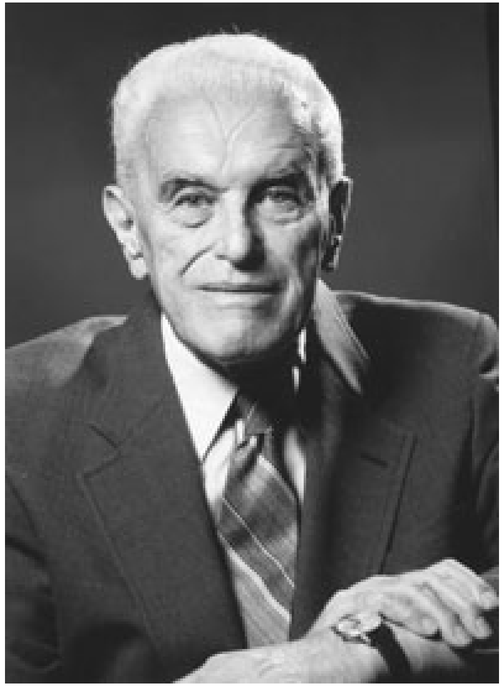
> "The purpose of this paper is to suggest an alternative approach to the analysis of games with incomplete information. This approach will be based on constructing, for any given game [of incomplete information], some game [of complete information] game-theoretically equivalent to [the first game]." (pp.164—165).
>
> "Thus, our approach will basically amount to replacing a game G involving incomplete information, by a new game $G^{\star}$ which involves complete but imperfect information, yet which is, as we shall argue, essentially equivalent to G from a game-theoretical point of view," (p.166).

Harsanyi, John C, 1976, "Games with Incomplete Information Played by 'Bayesian' Players I-III, Part I: The Basic Model," *Management Science* 14(3): 159—182

Asymmetric Information



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> "Accordingly, we define a [game of incomplete information] G where every player j *knows* the strategy spaces S_i of all players $i = 1, \cdots, j, \cdots n$ but where, in general, he does *not* know the payoff functions U_i of these players $i = 1, \cdots, j, \cdots n$," (p.166).

Harsanyi, John C, 1976, "Games with Incomplete Information Played by 'Bayesian' Players I-III, Part I: The Basic Model," *Management Science* 14(3): 159—182

Asymmetric Information

- Players have **beliefs** about other players' strategies & payoffs according to a probability distribution
    - Harsanyi shows it's sufficient to assume players have a "type"
- Shows that for every game of incomplete information, there are equivalent (sub-)games with complete (but imperfect) information
- "Bayesian" since players assumed to update their beliefs according to Bayes' rule (more on that next time)

Asymmetric Information

- Until now, our definition of a game (with complete information) has consisted of:

1. Players
2. Strategies
3. Payoffs (jointly determined by players' chosen strategies)

Asymmetric Information

- With a game of *incomplete* information a game consists of:

1. Players
   - **Types of players**
   - **Common prior beliefs about players**
2. Strategies
   - Strategies **conditioned on beliefs** about player types
3. Payoffs (jointly determined by players' chosen strategies)
   - **Payoffs depend on types**

Bayesian Simultaneous Games

- A new class of Bayesian games due to the role of information and beliefs
- We will consider simultaneous games first, then sequential games later
- Bayesian Nash equilibrium (BNE): set of strategies, one for each (**type** of) player where no (**type** of) player wants to change given what the others are doing
    - i.e. each (**type** of) player is playing a best response

Bayesian Nash Equilibria

- Two categories of equilibria in Bayesian games (with different players)

1. Pooling equilibrium: all types of players play the **same** strategy
2. Separating equilibrium: different types of players play **different** strategies

Bayesian Game Example

- Rowena and Colin play where they can each **Cooperate** or **Defect**

- Suppose Colin could be one of two types:

  - **Prisoners' Dilemma**-type payoffs:

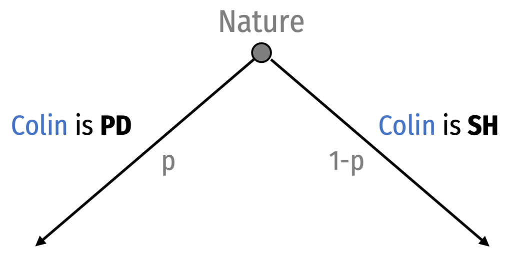
$$(C, D) \succ (C, C) \succ (D, D) \succ (D, C)$$

  - **Stag Hunt**-type payoffs:

$$(C, C) \succ (D, D) \sim (C, D) \succ (D, C)$$

- Rowena has **Stag Hunt**-type payoffs

$$(C, C) \succ (D, D) \sim (D, C) \succ (C, D)$$



Colin is PD:

Rowena \ Colin	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

Colin is SH:

Rowena \ Colin	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

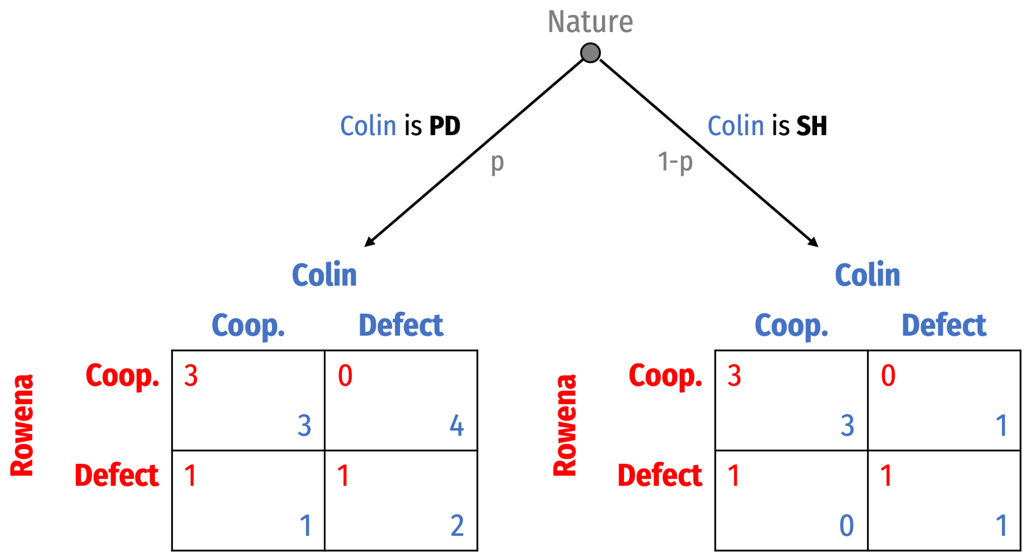
Interpreting Nature in the Example

1. The identity of the other player is known, but **their preferences are unknown**
    - Rowena know she is playing with Colin, but doesn't know if he has PD or SH preferences
    - **Nature** whispers to Colin his type, and Rowena has to figure it out

2. Nature selects Colin from a population of **potential player types**
    - Rowena knows she will play with another player, but doesn't know if s/he is playing PD or SH, **Nature** decides

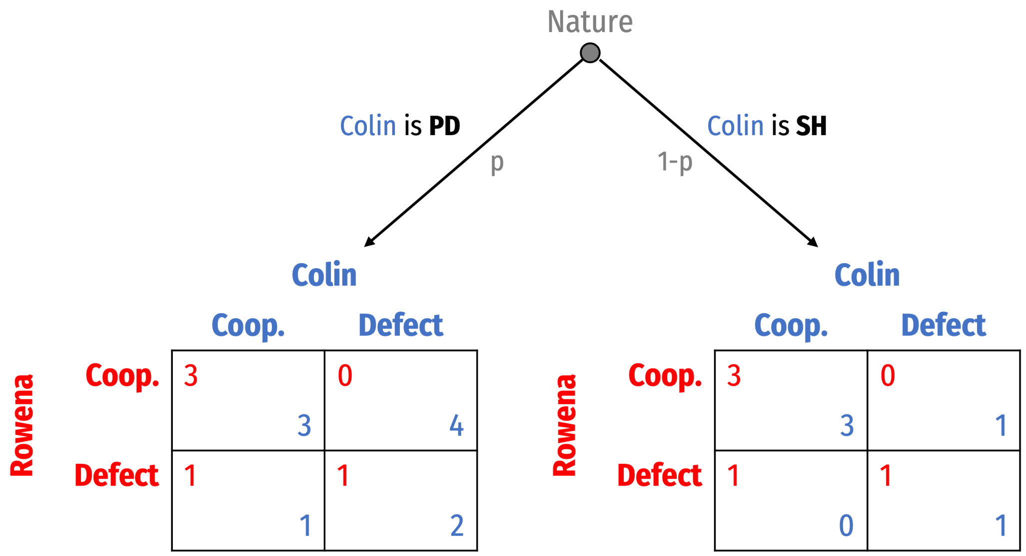
Bayesian Game Example

- Rowena must choose between two strategies, Cooperate or Defect



Bayesian Game Example

- Rowena must choose between two strategies, Cooperate or Defect
- If she plays Cooperate
  - A **SH**-type Colin will want to Cooperate, both get 3
  - A **PD**-type Colin will want to Defect, giving her 1
- Rowena will have to consider her own **expected payoff** of playing each strategy against both types of Colin
  - Depends on her beliefs about p!



Bayesian Game Example

- Simple example, suppose (she believes) $p = 1.00$, Rowena is for sure playing against a **PD**-type Colin
- Game simplifies to a game of **complete imperfect** information
- **Pure strategy Nash equilibrium**: (Defect, Defect)
  - Colin has dominant strategy to Defect
  - Rowena's best response is to Defect

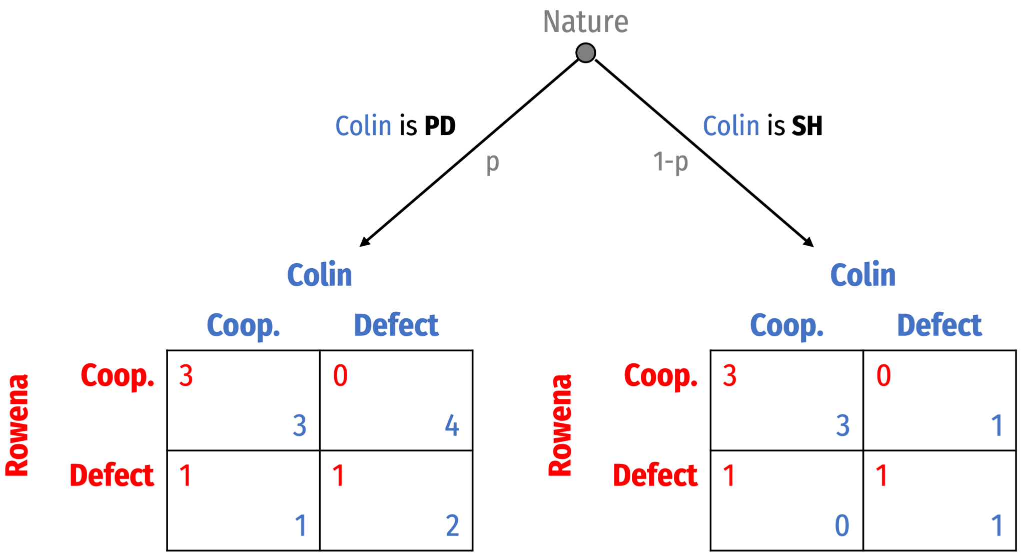
Rowena \ Colin	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

Exploring BNE in Bayesian Game Example

- Different *potential* **Bayesian Nash equilibrium (BNE)**:

1) **Pooling equilibria**: both types of Colin play the same strategy

- **Scenario I**: Colin-types both Cooperate
- **Scenario II**: Colin-types both Defect

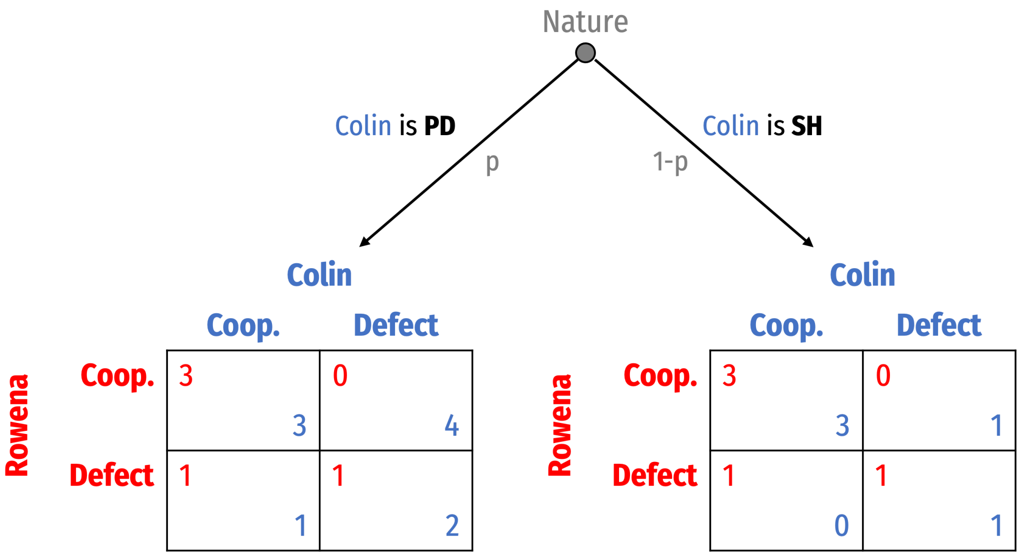


Exploring BNE in Bayesian Game Example

- Different *potential* **Bayesian Nash equilibrium (BNE)**:

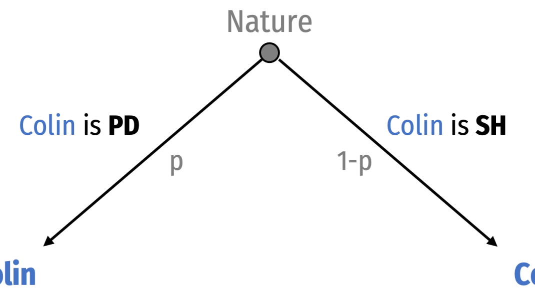
2) **Separating equilibria**: each type of Colin plays a different strategy

- **Scenario III**: PD-type Colin plays Cooperate; SH-type Colin plays Defect
- **Scenario IV**: PD-type Colin plays Defect; SH-type Colin plays Cooperate



Pooling Eq. I: Both Types Cooperate (?)

- **Pooling equilibrium I**: both Colin-types play Cooperate, i.e. (C,C)
- ❌ This is **impossible**: PD-type Colin has a dominant strategy to Defect
  - Would switch from Cooperate to Defect



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Rowena \ Colin	Coop.	Defect
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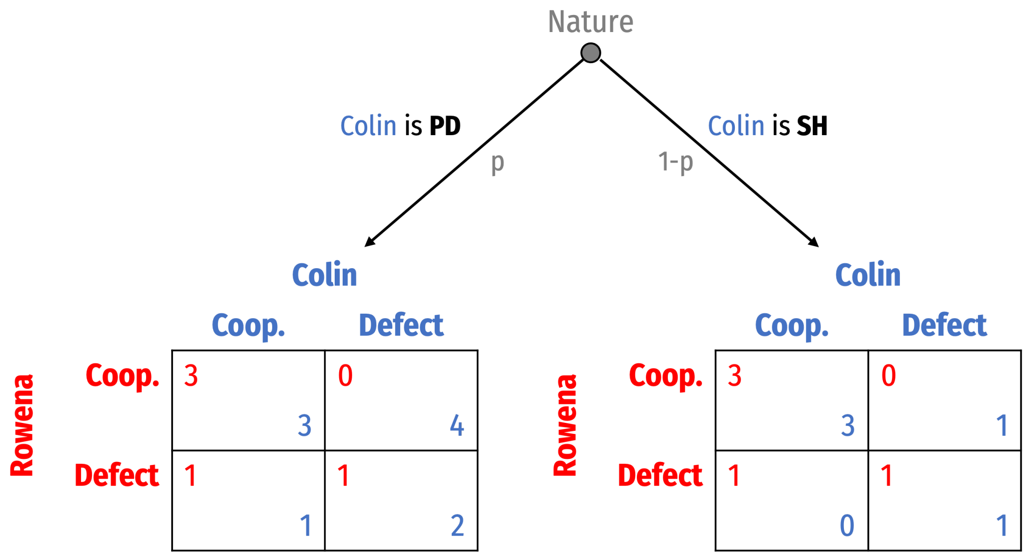
Defect	1, 1	1, 2

Colin is SH:

Rowena \ Colin	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Pooling Eq. II: Both Types Defect (?)

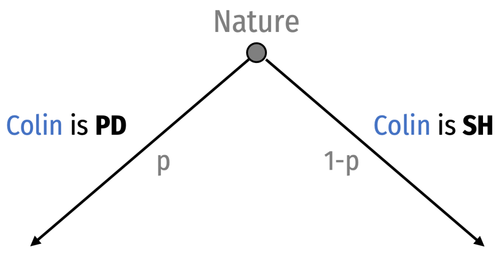
- Pooling equilibrium II: both Colin-types play Defect, i.e. (D,D)
- Rowena maximizes her *expected* payoff against unknown Colin-type playing Defect



Pooling Eq. II: Both Types Defect (?)

- Pooling equilibrium II: both Colin-types play Defect, i.e. (D,D)
- Rowena maximizes her *expected* payoff against unknown Colin-type playing Defect

$$E[Cooperate] = 0p + 0(1-p)$$
$$E[Cooperate] = 0$$



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Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Pooling Eq. II: Both Types Defect (?)

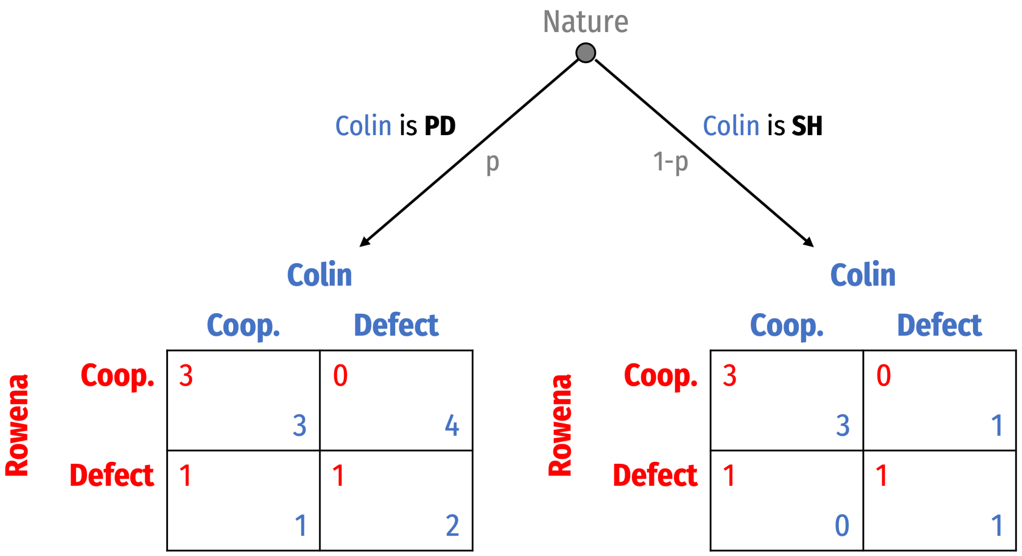
- Pooling equilibrium II: both Colin-types play Defect, i.e. (D,D)

- Rowena maximizes her *expected* payoff against unknown Colin-type playing Defect

$$E[Cooperate] = 0p + 0(1-p)$$
$$E[Cooperate] = 0$$

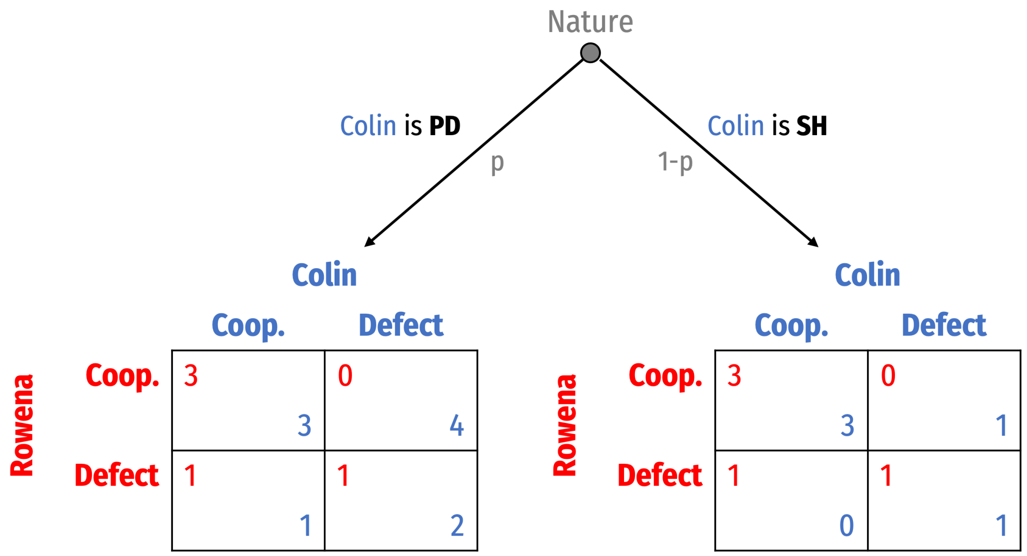
$$E[Defect] = 1p + 1(1-p)$$
$$E[Defect] = 1$$

- Rowena will always play Defect



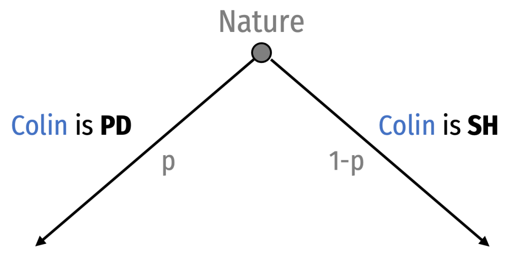
Pooling Eq. II: Both Types Defect (?)

- **Pooling equilibrium II**: both Colin-types play Defect. i.e. (D,D)
- ✅ This is a valid **Bayesian Nash Equilibrium**: {Defect, (Defect, Defect)}
    - Where Colin's strategies are denoted for (PH-type Colin, SH-type Colin)



Separating Eq. I: PD-Type Coops; SH-Type Defects (?)

- **Separating equilibrium I**: PD-type Colin plays Cooperate; SH-type Colin plays Defect, i.e. (C,D)



Colin is PD (Rowena payoff, Colin payoff)

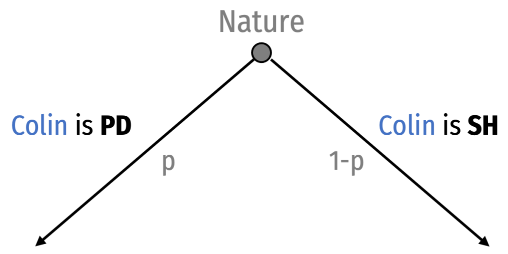
Rowena \ Colin	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

Colin is SH (Rowena payoff, Colin payoff)

Rowena \ Colin	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Separating Eq. I: PD-Type Coops; SH-Type Defects (?)

- **Separating equilibrium I**: PD-type Colin plays Cooperate; SH-type Colin plays Defect, i.e. (C,D)
- ❌ This is **impossible**: PD-type Colin has a dominant strategy to Defect
  - Would switch from Cooperate to Defect



Colin

Rowena	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

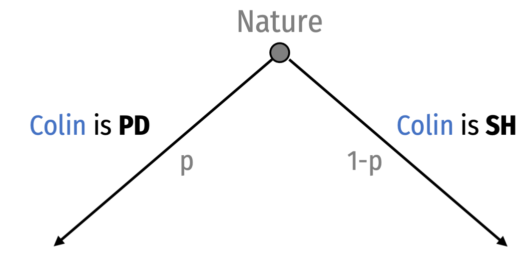
Colin

Rowena	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Separating Eq. II: PD-Type Defects; SH-Type Coops. (?)

- **Separating equilibrium II**: PD-type Colin plays Defect; SH-type Colin plays Cooperate, i.e. (D,C)

- Rowena maximizes her *expected* payoff against unknown Colin-type:



Colin (Colin is PD)

Rowena	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

Colin (Colin is SH)

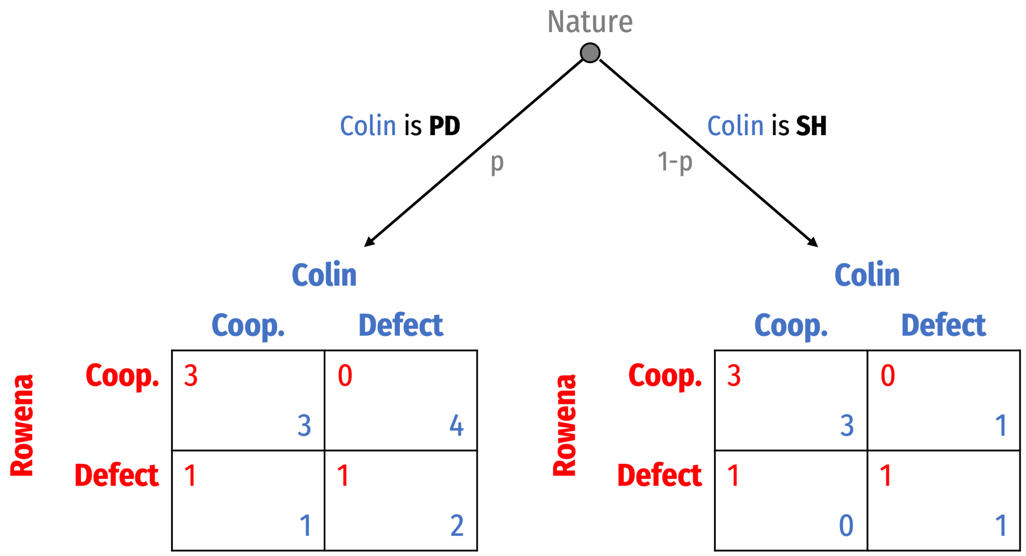
Rowena	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Separating Eq. II: PD-Type Defects; SH-Type Coops. (?)

- **Separating equilibrium II**: PD-type Colin plays Defect; SH-type Colin plays Cooperate, i.e. (D,C)

- Rowena maximizes her *expected* payoff against unknown Colin-type:

$$E[Cooperate] = 0p + 3(1-p)$$
$$E[Cooperate] = 3 - 3p$$



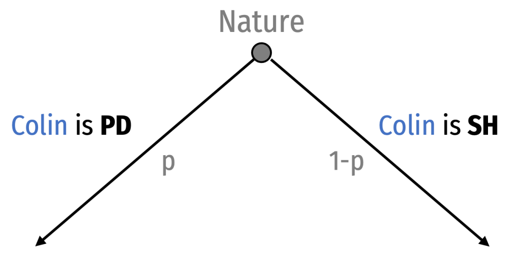
Separating Eq. II: PD-Type Defects; SH-Type Coops. (?)

- **Separating equilibrium II**: PD-type Colin plays Defect; SH-type Colin plays Cooperate, i.e. (D,C)

- Rowena maximizes her *expected* payoff against unknown Colin-type:

$$E[Cooperate] = 0p + 3(1-p)$$
$$E[Cooperate] = 3 - 3p$$

$$E[Defect] = 1p + 1(1-p)$$
$$E[Defect] = 1$$



Colin (PD)

Rowena	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

Colin (SH)

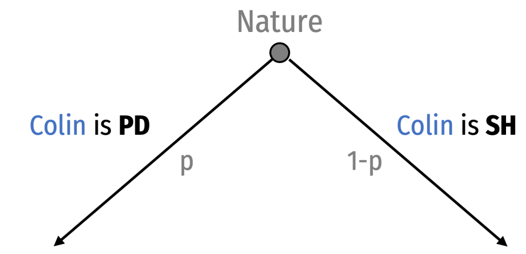
Rowena	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Separating Eq. II: PD-Type Defects; SH-Type Coops. (?)

- **Separating equilibrium II**: PD-type Colin plays Defect; SH-type Colin plays Cooperate, i.e. (D,C)

- Rowena maximizes her *expected* payoff against unknown Colin-type:

$$E[Cooperate] = E[Defect]$$
$$3 - 3p = 1$$
$$p = \frac{2}{3}$$



Colin (Colin is PD)

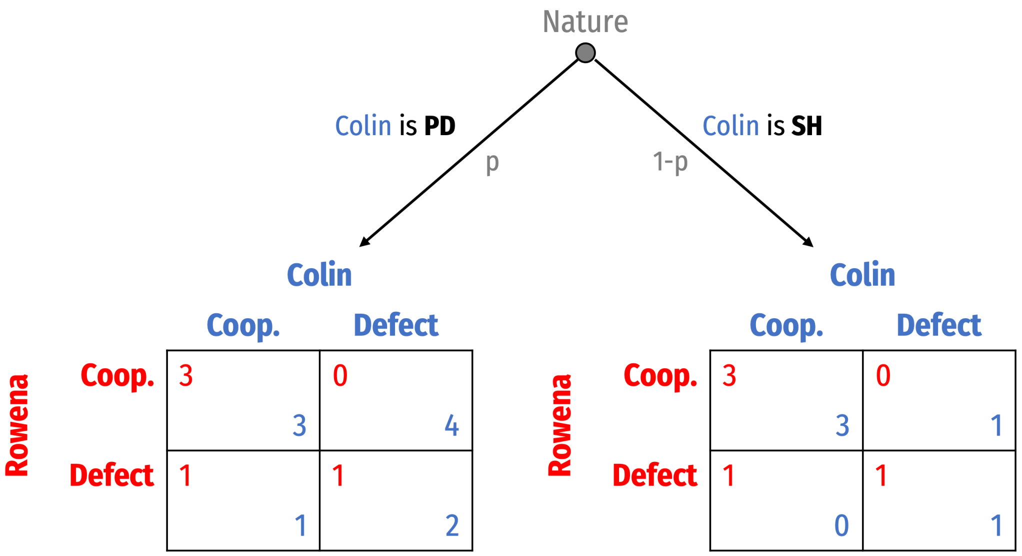
Rowena	Coop.	Defect
Coop.	3, 3	0, 4
Defect	1, 1	1, 2

Colin (Colin is SH)

Rowena	Coop.	Defect
Coop.	3, 3	0, 1
Defect	1, 0	1, 1

Separating Eq. II: PD-Type Defects; SH-Type Coops. (?)

- **Separating equilibrium II**: PD-type Colin plays Defect; SH-type Colin plays Cooperate, i.e. (D,C)
- When $p > \frac{2}{3}$, Rowena should play Defect ✅
- When $p < \frac{2}{3}$, Rowena should play Cooperate
  - Here, a SH-type Colin would want to switch from Cooperate to Defect; this would not be a BNE ❌



Bayesian Nash Equilibria

- Our two possible **Bayesian Nash equilibria (BNE)**:

1. {Defect, (Defect, Defect)}, a **pooling equilibrium**

2. {Defect, (Cooperate, Defect)}, a **separating equilibrium**, if $p > \frac{2}{3}$

    - If $p < \frac{2}{3}$, no equilibrium

- Note that these depend on Rowena's **beliefs** about p!
    - Next…why these are **Bayesian** games