# 4.1 — Subgame Perfection

## ECON 316 • Game Theory • Fall 2021


Ryan Safner

Assistant Professor of Economics

safner@hood.edu

ryansafner/gameF21

gameF21.classes.ryansafner.com

# Outline

# Subgame Perfection

# A Motivating Example

- Suppose I announce that if any of you were late, I would give you an F
- If you believe my threat, you will arrive on time, and I never have to carry out my threat
- *Sounds* like a Nash equilibrium:
    - I get what I want at no cost to me
    - You prefer being in class on time to failing
    - Nobody wants to change

# A Motivating Example

- Implausible prediction: I would not actually want to carry out my threat if it came to it!
  - Big confrontation, you could complain to Dept. chair, Provost, etc
- A problem of “out-of-equilibrium” play
  - How can a threat *I will never carry out* change your behavior?
  - I can optimally choose bizarre behavior in situations I know will never happen!

# A Motivating Example

- BUT: if you know what *would* happen in those unlikely scenarios, that *does* affect your behavior for things that *normally* happen
    - namely, if you know I will not *actually* fail you for coming late, you will sometimes come late

# Motivating Example

- This lesson is about the effects of **threats** and **promises**
- Must learn another major refinement of Nash equilibrium
- First, return to seqential games
- Continue with assumption of perfect information (soon we will consider imperfect information)

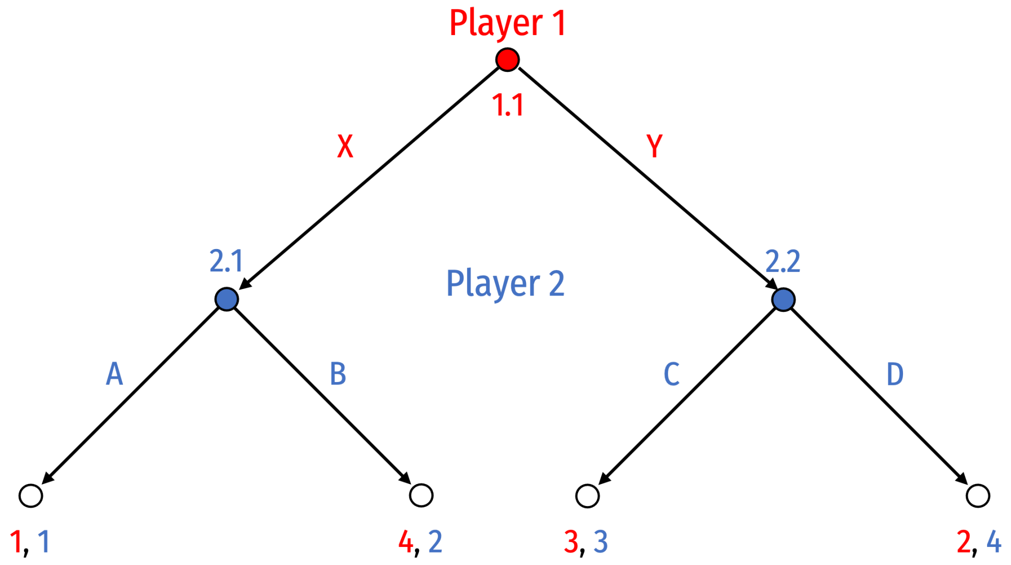
# Motivating Example

- A new solution concept:
- **Subgame perfect Nash equilibrium (SPNE)**: selects only Nash equilibria sustained by **credible** threats and promises, and rules out *non-credible* threats/promises
  - Formal definition: a set of strategies is **SP** if it induces a Nash equilibrium in *every subgame* of a game
- First, let's understand what we mean by "subgame"

# Subgames

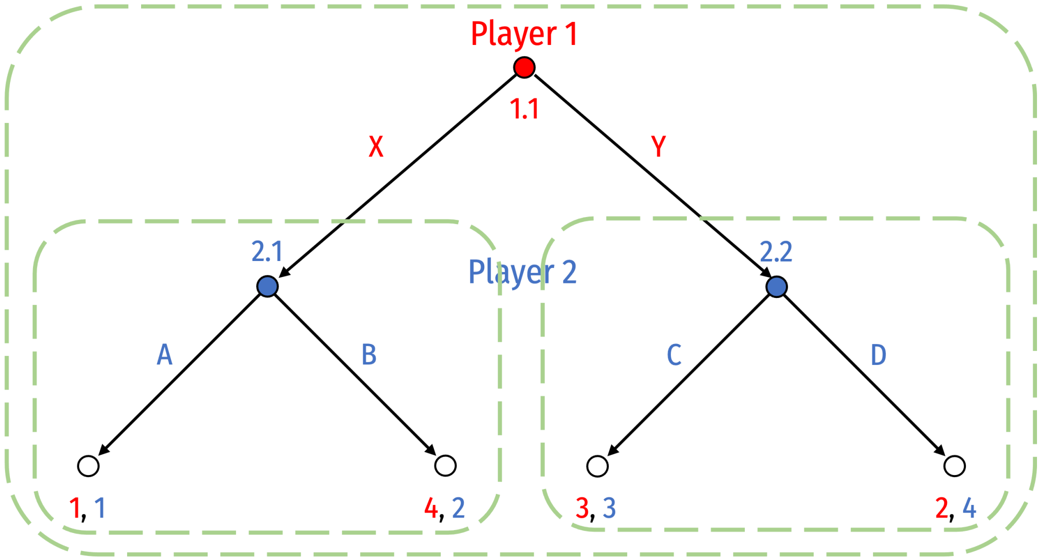
# Subgames

- A **subgame** is any portion of a game that contains one initial note and all of its successor nodes
    - e.g. **any decision node initiates its own subgame** through to the terminal nodes
    - The game itself counts as a subgame
- Idea: analyze a subgame as a game itself and **ignore any history** in the overall game and **find what is optimal in each subgame**



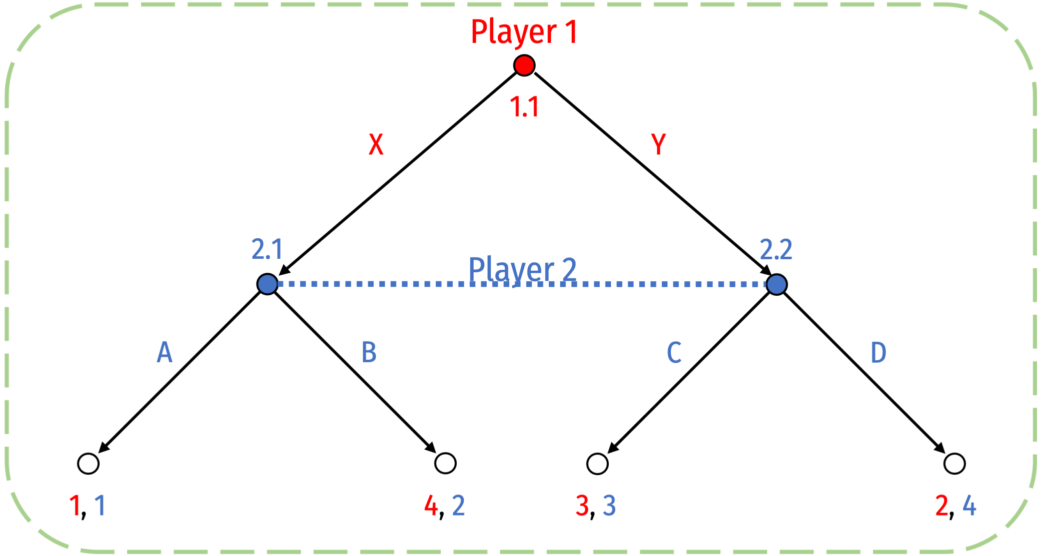
# Subgames: Example

- In this example, there are 3 subgames:
    1. The full game itself (initiated by Player 1's decision node 1.1)
    2. Subgame initiated by Player 2's decision node 2.1
    3. Subgame initiated by Player 2's decision node 2.2



# Aside: Subgames Can't Break Information Sets

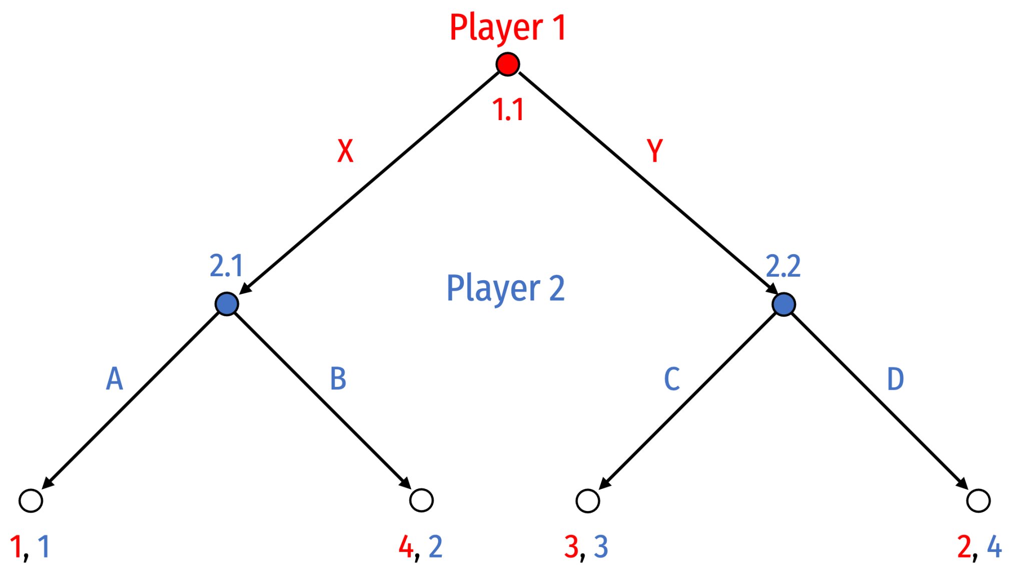
- Subgames cannot “break” **information sets**
    - Indicated by dashed line: Player 2 does not know what Player 1 chose (consider it a simultaneous game)
    - More on information later
- Players must *know which* subgame they are in, so a subgame cannot “break” an information set
    - Player 2 here would not know what Player 1 did, so Player 1 can’t make a decision; could not “ignore history”



# (Review) Strategies in this Example

- Recall we defined a **strategy** as a complete plan of what a player will do at *every* decision node they (might) face
- Player 1 has 1 decision (1.1) with 2 choices, so $2^1$ possible strategies:

  1. X at (1.1)
  2. Y at (1.1)



# (Review) Strategies in this Example

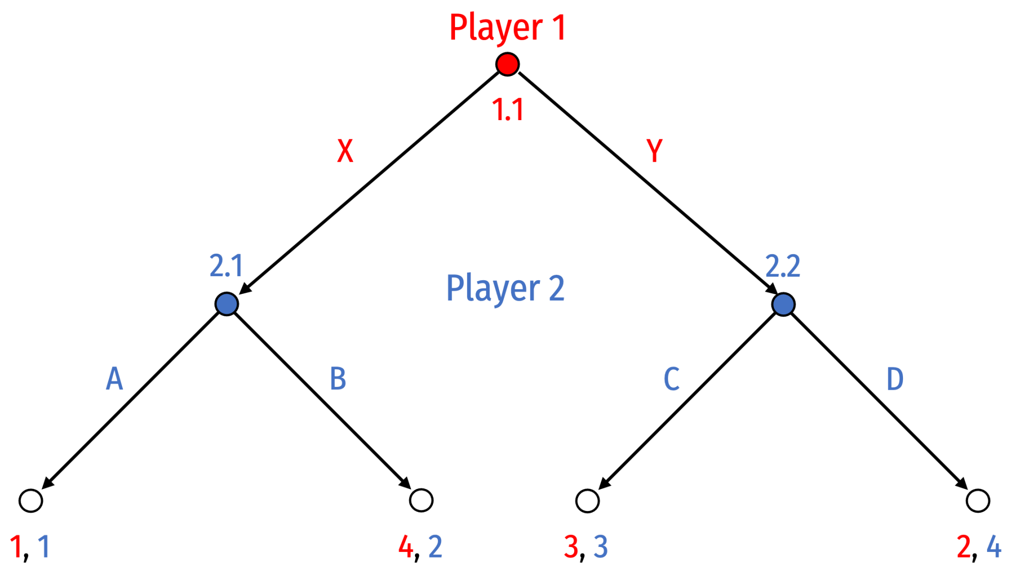
- Recall we defined a **strategy** as a complete plan of what a player will do at *every* decision node they (might) face

- Player 1 has 1 decision (1.1) with 2 choices, so $2^1$ possible strategies:

  1. X at (1.1)
  2. Y at (1.1)

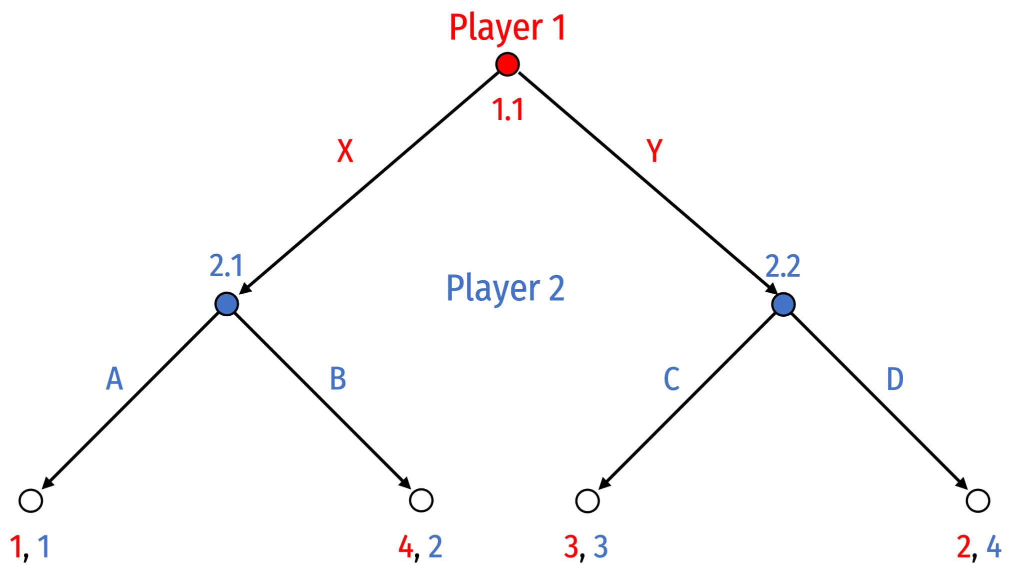
- Player 2 has 2 decision (2.1, 2.2) with 2 choices at each, so $2^2$ possible strategies:

  1. A at (2.1); C at (2.2)
  2. A at (2.1); D at (2.2)
  3. B at (2.1); C at (2.2)



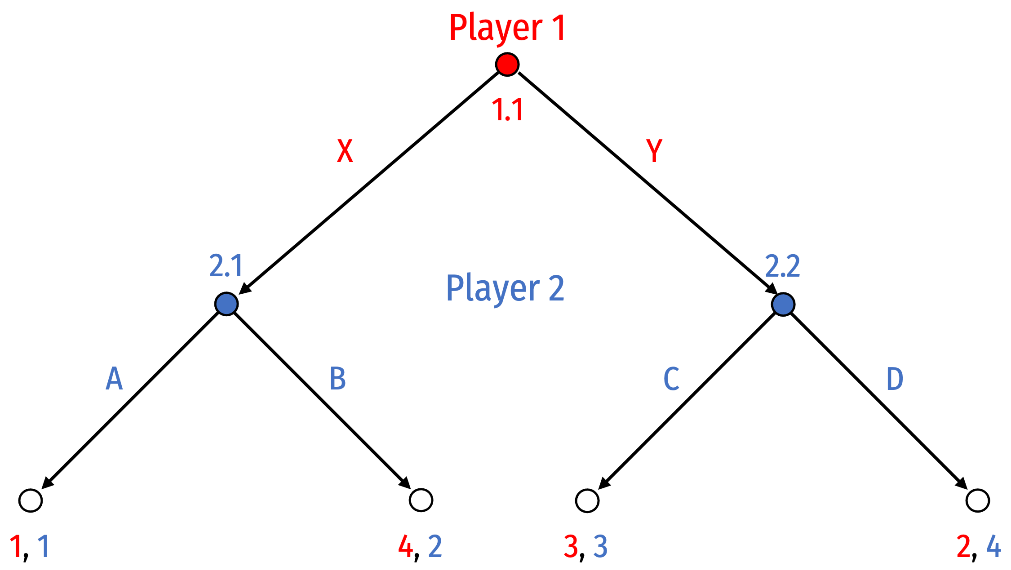
# Converting Between Sequential and Normal Form

- We can convert any sequential game in extended form (game tree) into a normal game (payoff matrix)
  - Harder to go the other way around!



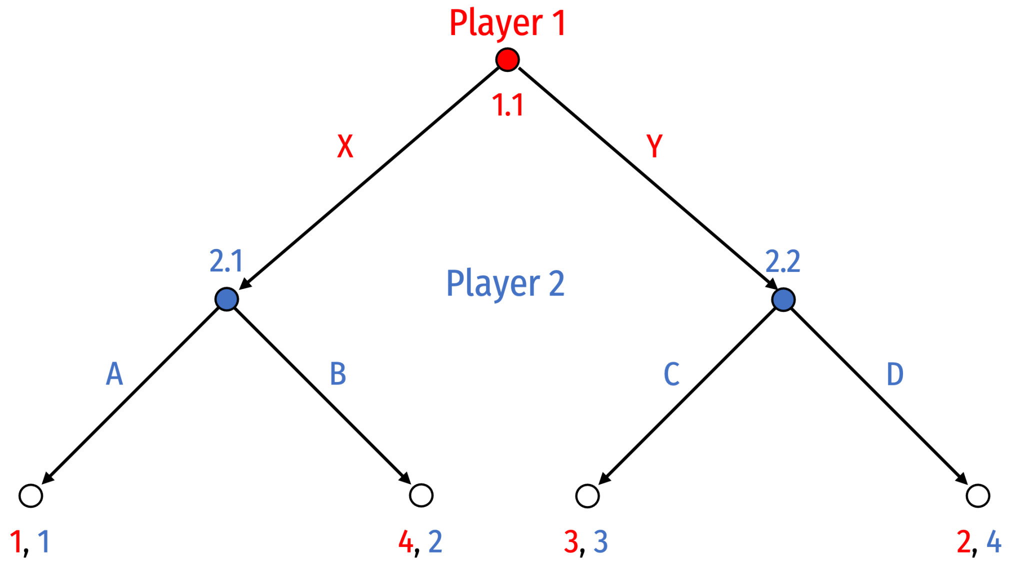
# Converting Between Sequential and Normal Form

- We can convert any sequential game in extended form (game tree) into a normal game (payoff matrix)
  - Harder to go the other way around!
- Payoff matrix of outcomes of all possible combinations of strategies for each player



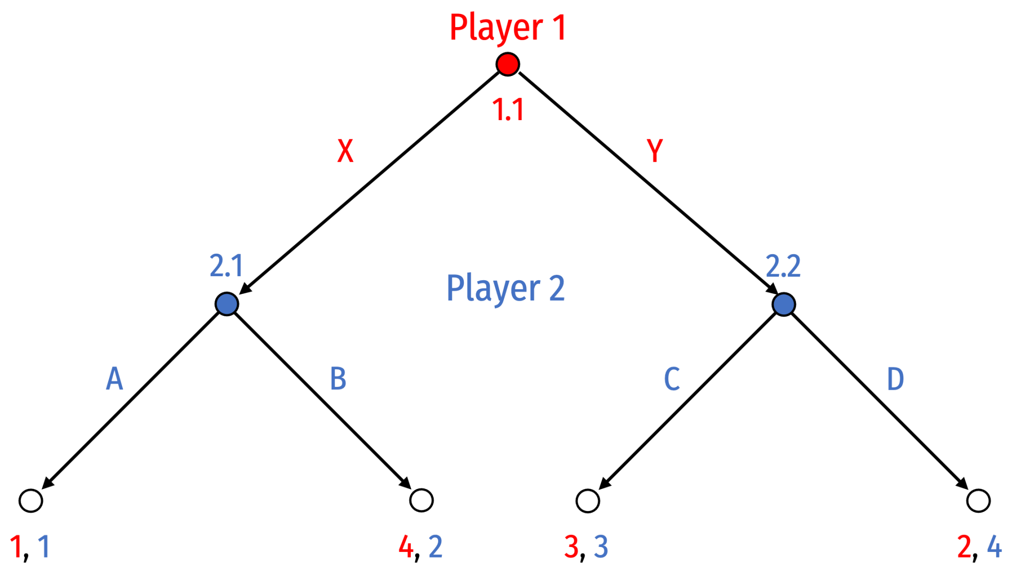
# Converting Between Sequential and Normal Form

- Solve the normal form for Nash equilibria



# Converting Between Sequential and Normal Form

- Nash equilibria:
  1. {Y, (A,D)}
  2. {X, (B,C)}
  3. {X, (B,D)}

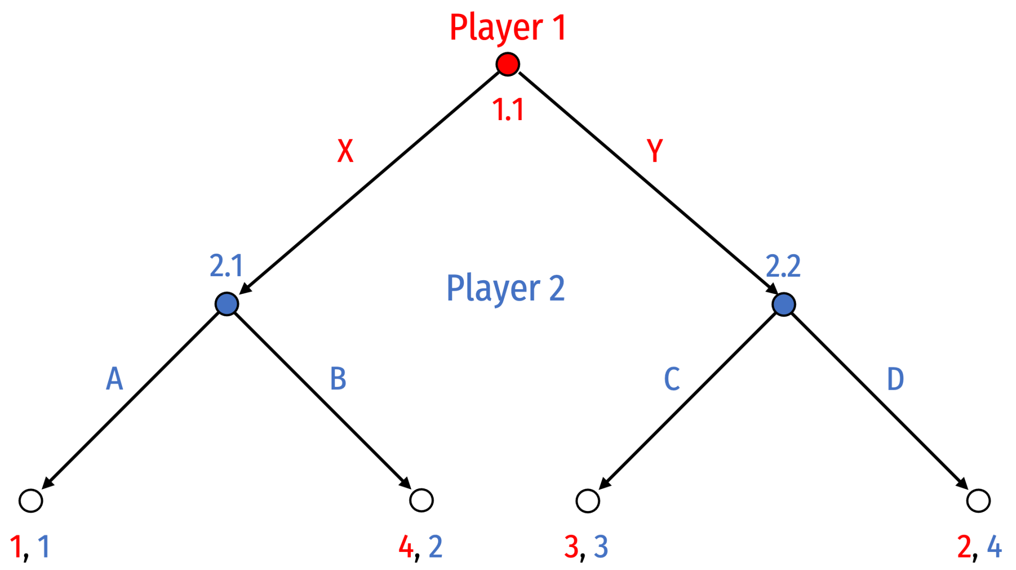


# Converting Between Sequential and Normal Form

- Nash equilibria:

  1. {Y, (A,D)}
  2. {X, (B,C)}
  3. {X, (B,D)}

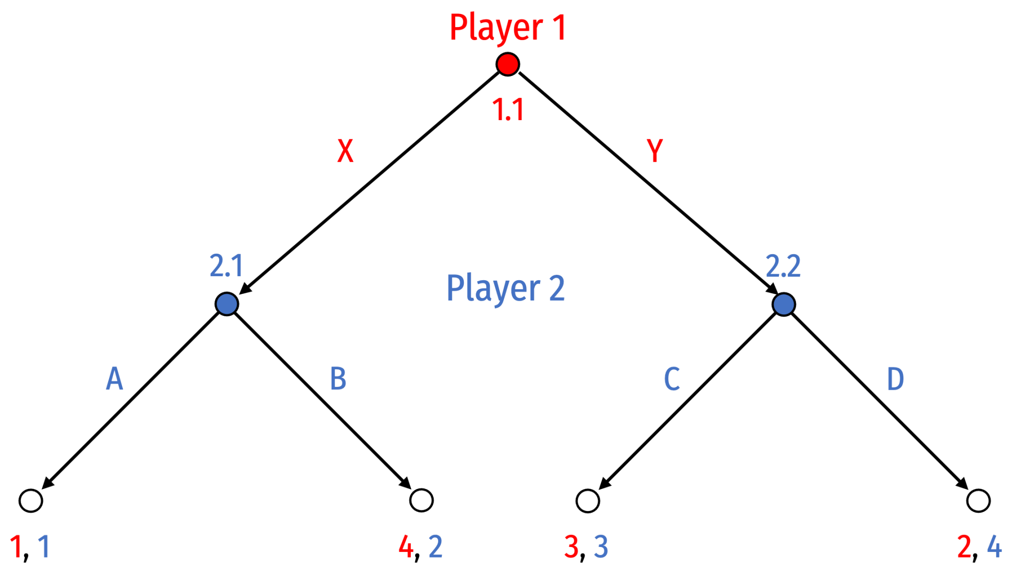
- But remember, this is a sequential game! Which of these Nash equilibria is **sequentially-rational**?



# Rollback Equilibrium

- Solve for rollback equilibrium via backwards induction
- A process of considering **"sequential rationality"**:

> "If I play x, my opponent will respond with y; given their response, do I really want to play x? …"

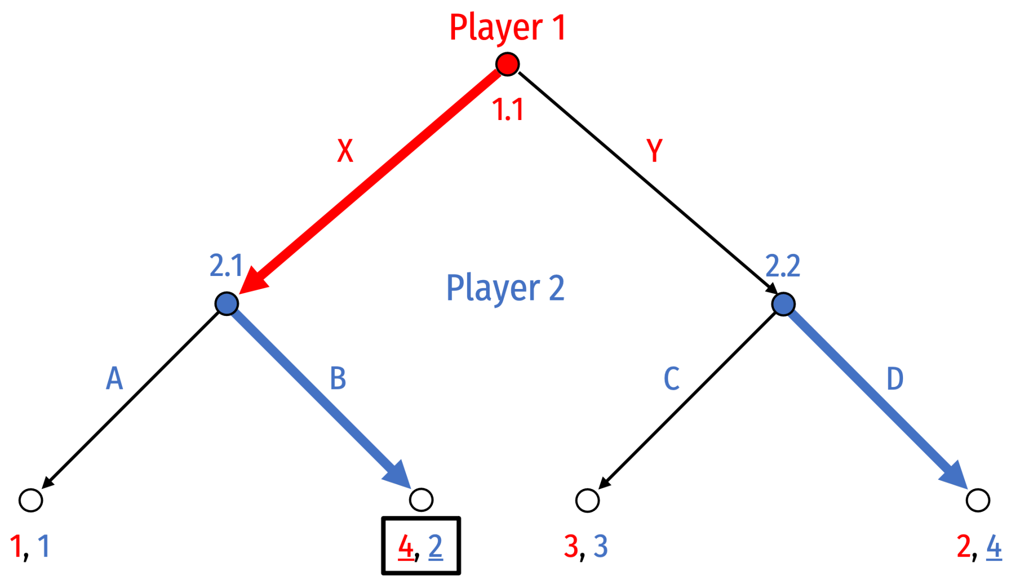


# Converting Between Sequential and Normal Form

- Nash equilibria:

    1. {Y, (A,D)}
    2. {X, (B,C)}
    3. {X, (B,D)}

- **Rollback equilibrium:** {X, (B,D)}



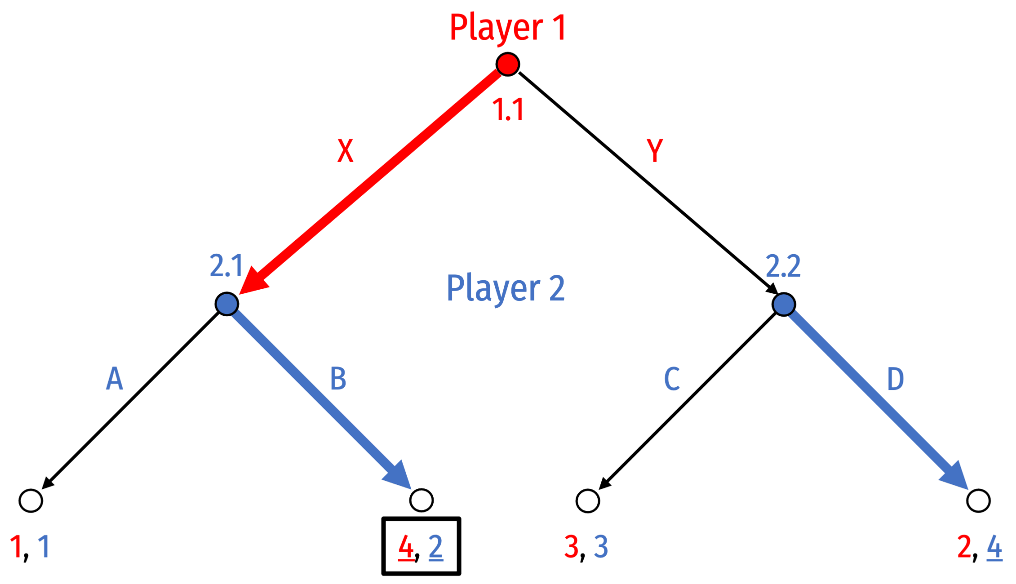
# Converting Between Sequential and Normal Form

- Nash equilibria:

  1. {Y, (A,D)}
  2. {X, (B,C)}
  3. {X, (B,D)}

- Even though there are three Nash equilibria, only one is **subgame perfect**

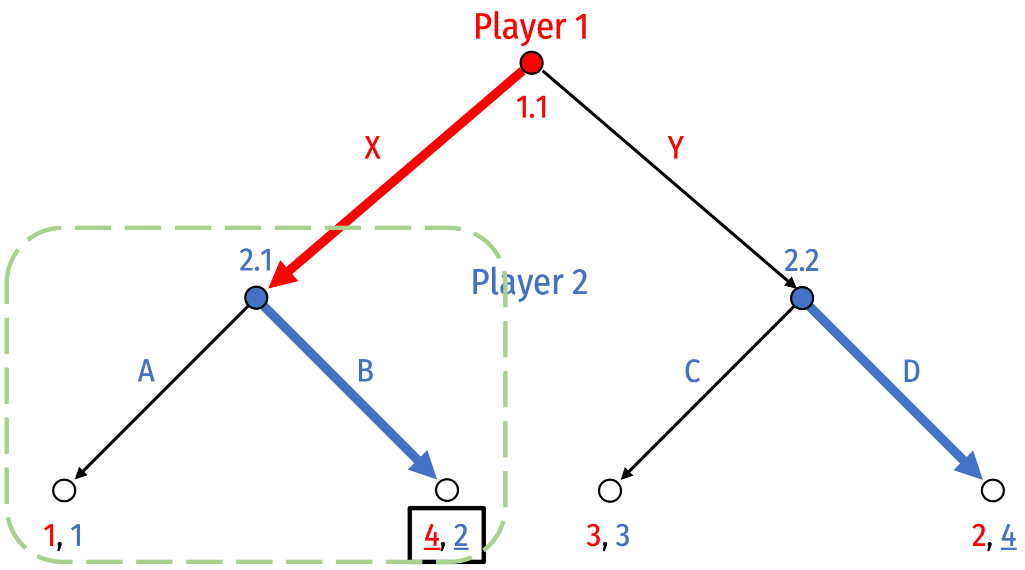
  - Player 1 and Player 2 are playing {X, (B,D)} respectively causes a **Nash equilibrium in <u>every</u> subgame**



# Converting Between Sequential and Normal Form

- Nash equilibria:
  1. {Y, (A,D)}
  2. {X, (B,C)}
  3. {X, (B,D)}

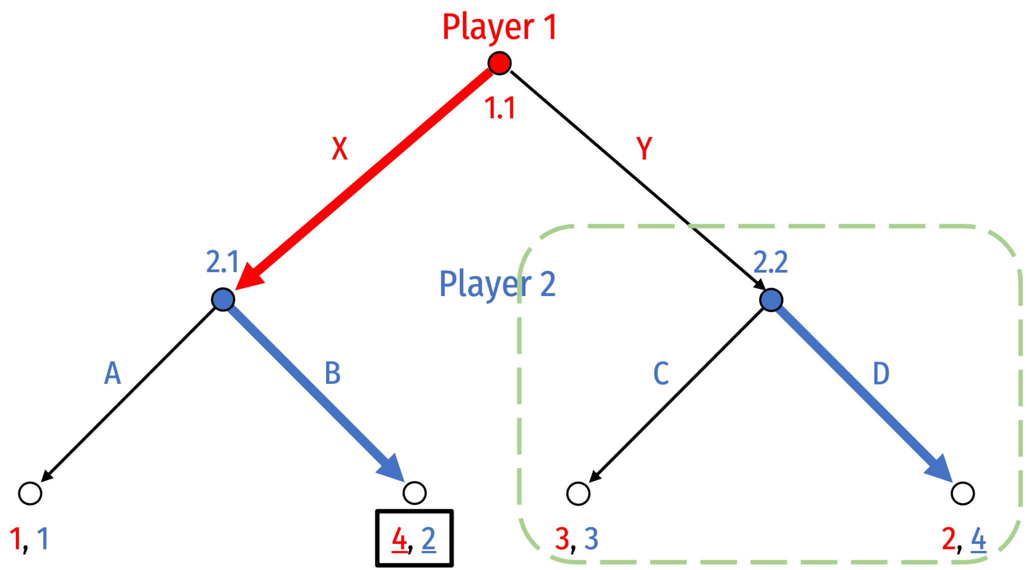
- Consider the first NE: {Y, (A,D)}
  - Not on the equilibrium path of play
  - Not sequentially rational: if Player 1 had played X (for whatever reason), Player 2 would want to switch from playing A to playing B at 2.1!
  - Thus, this strategy is not a NE in subgame initiated at node 2.1 (Player 2 would want to change strategies)



# Converting Between Sequential and Normal Form

- Nash equilibria:
  1. {Y, (A,D)}
  2. {X, (B,C)}
  3. {X, (B,D)}

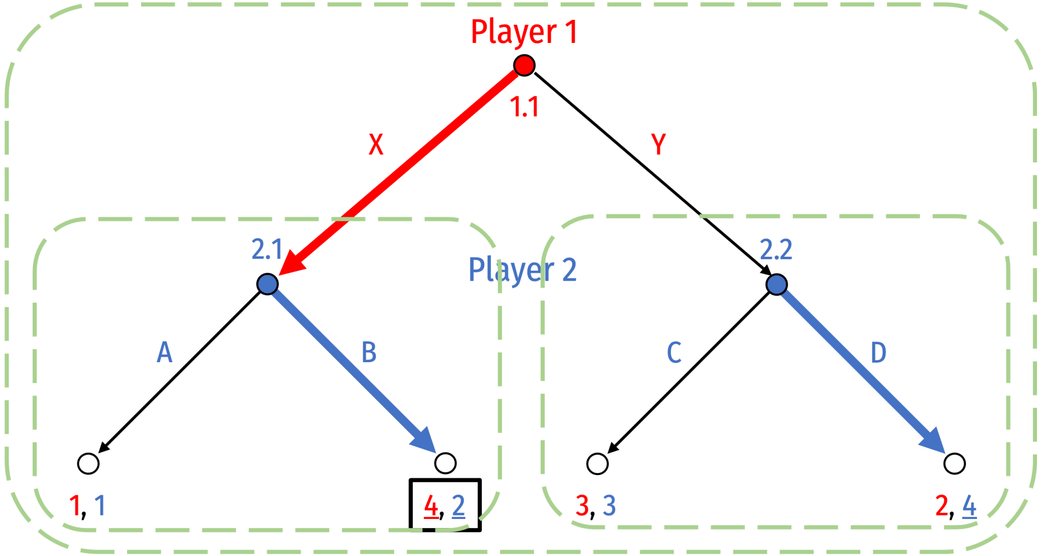
- Consider the second NE: {X, (B,C)}
  - Not on the equilibrium path of play
  - Not sequentially rational: if Player 1 had played Y (for whatever reason), Player 2 would want to switch from playing C to playing D at 2.2!
  - Thus, this strategy is not a NE in subgame initiated at node 2.2 (Player 2 would want to change strategies)



# Converting Between Sequential and Normal Form

- Nash equilibria:
  1. {Y, (A,D)}
  2. {X, (B,C)}
  3. {X, (B,D)}

- Consider the third NE: {X, (B,D)}
  - On the equilibrium path of play
  - Sequentially rational: these strategies lead to a NE in *every* subgame!
  - Conveniently: **the "rollback equilibrium" is always subgame perfect**

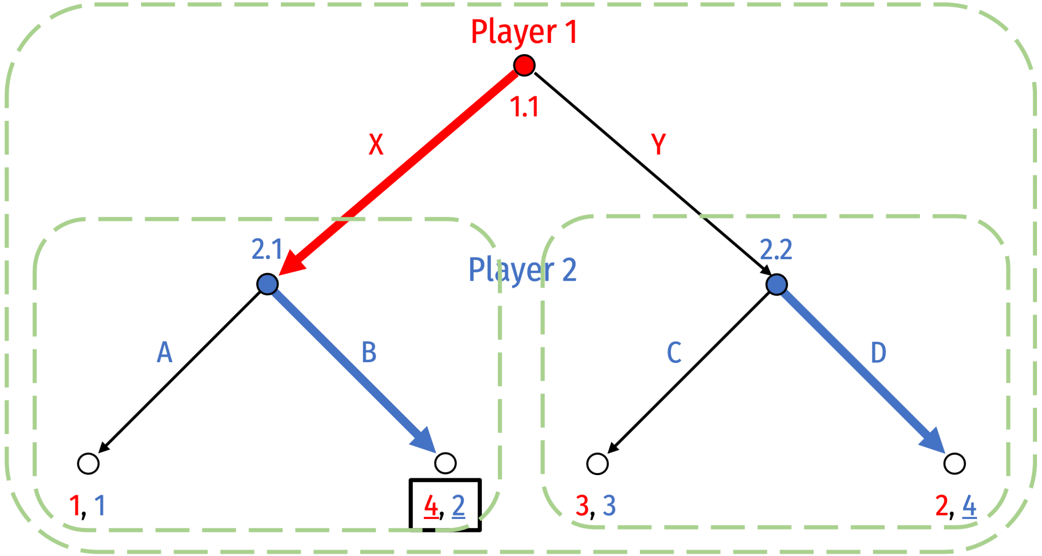



**Player 2** / **Player 1**

| | (A,C) | (A,D) | (B,C) | (B,D) |
|---|---|---|---|---|
| **X** | 1, 1 | 1, 1 | 4, 2 | 4, 2 |
| **Y** | 3, 3 | 2, 4 | 3, 3 | 2, 4 |

# Converting Between Sequential and Normal Form

- Subgame perfection rules out **non-credible threats or promises**
- Depending on context, Player 2 might threaten/promise that they will play C if Player 1 plays Y
  - But if that subgame were reached, Player 2 would *not* play C, they would want to play D!
  - i.e. not a credible claim



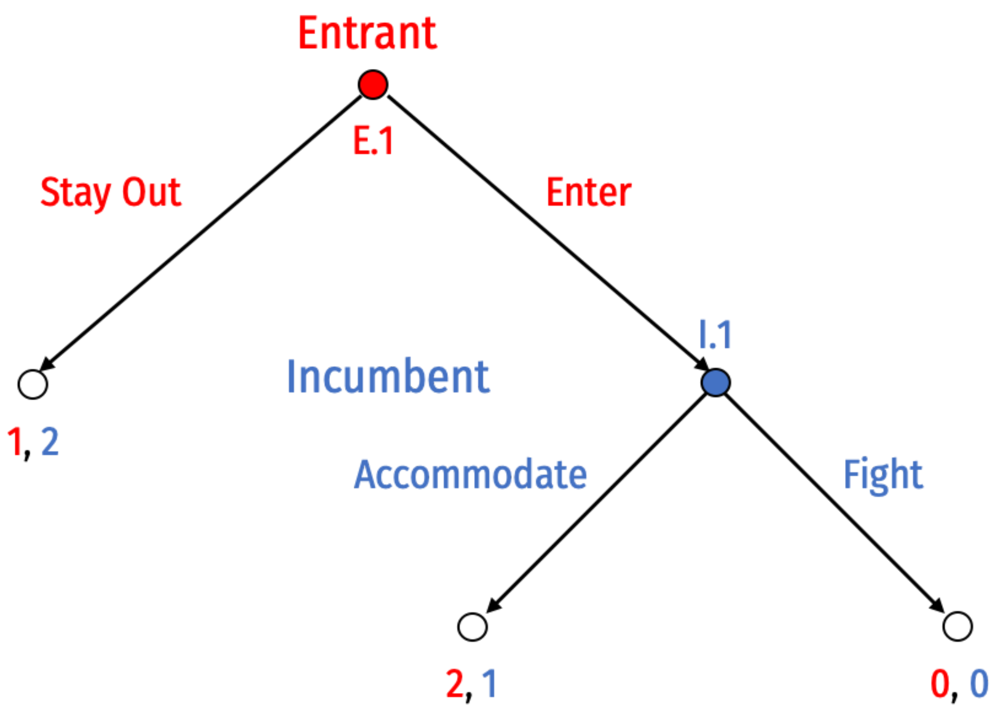

**Player 2**

| Player 1 | | (A,C) | (A,D) | (B,C) | (B,D) |
|---|---|---|---|---|---|
| | **X** | 1, 1 | 1, 1 | 4, 2 | 4, 2 |
| | **Y** | 3, 3 | 2, 4 | 3, 3 | 2, 4 |

# Entry Game Example

# Entry Game: Extensive Form

- Consider an Entry Game, a sequential game played between a potential Entrant and an Incumbent



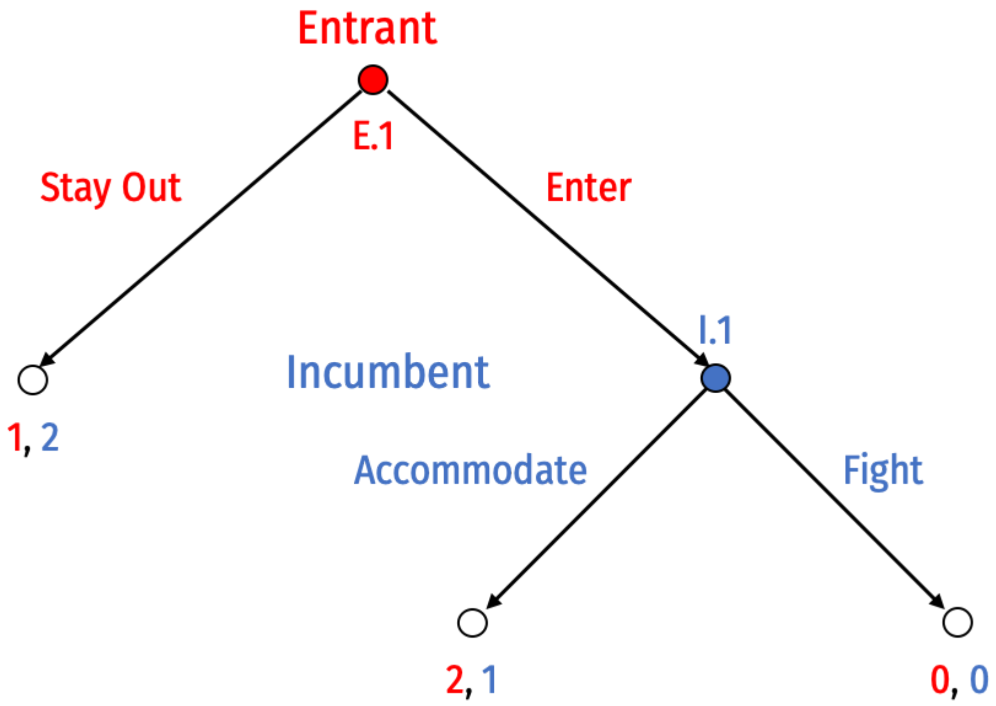
# Entry Game: (Pure) Strategies

- **Entrant** has 2 pure strategies:

  1. Stay Out at E.1
  2. Enter at E.1

- **Incumbent** has 2 pure strategies:

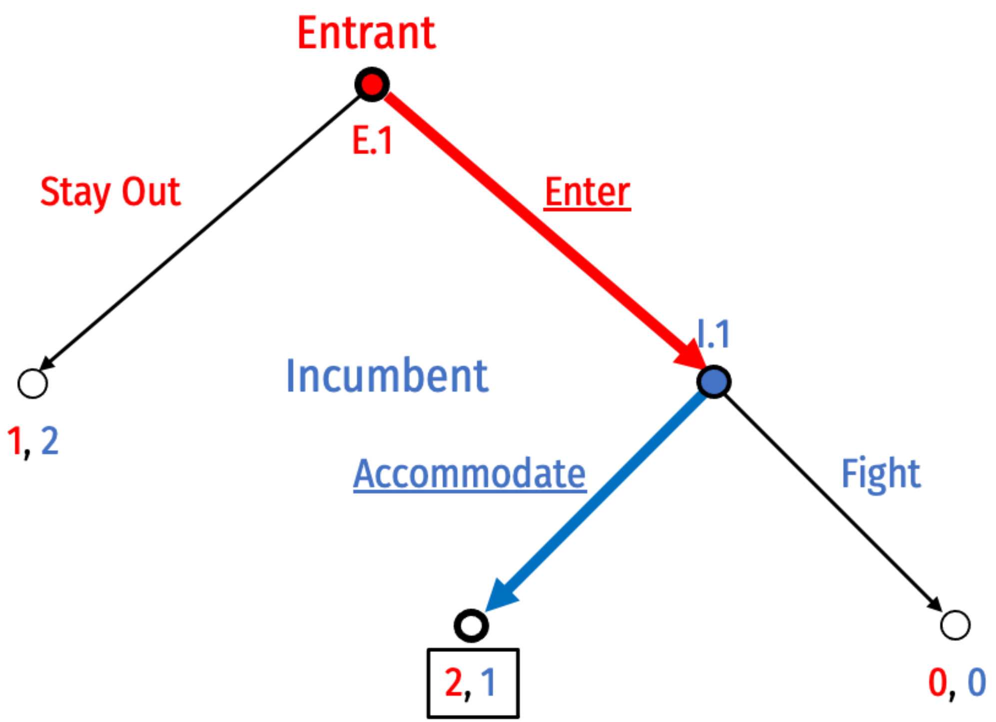
  1. Accommodate at I.1
  2. Fight at I.1



# Entry Game: Backward Induction

- Rollback/Subgame Perfect Nash Equilibrium:

  (Enter, Accommodate)



# Entry Game: Normal vs. Extensive Form

- Convert this game to Normal form
- Note, if **Entrant** plays Stay Out, doesn't matter what **Incumbent** plays, payoffs are the same
- Solve this for Nash Equilibria...

| | | Incumbent | |
|---|---|---|---|
| | | **Accommodate** | **Fight** |
| **Entrant** | **Enter** | 2, 1 | 0, 0 |
| | **Stay Out** | 1, 2 | 1, 2 |

# Entry Game: Normal vs. Extensive Form

- Two Nash Equilibria:

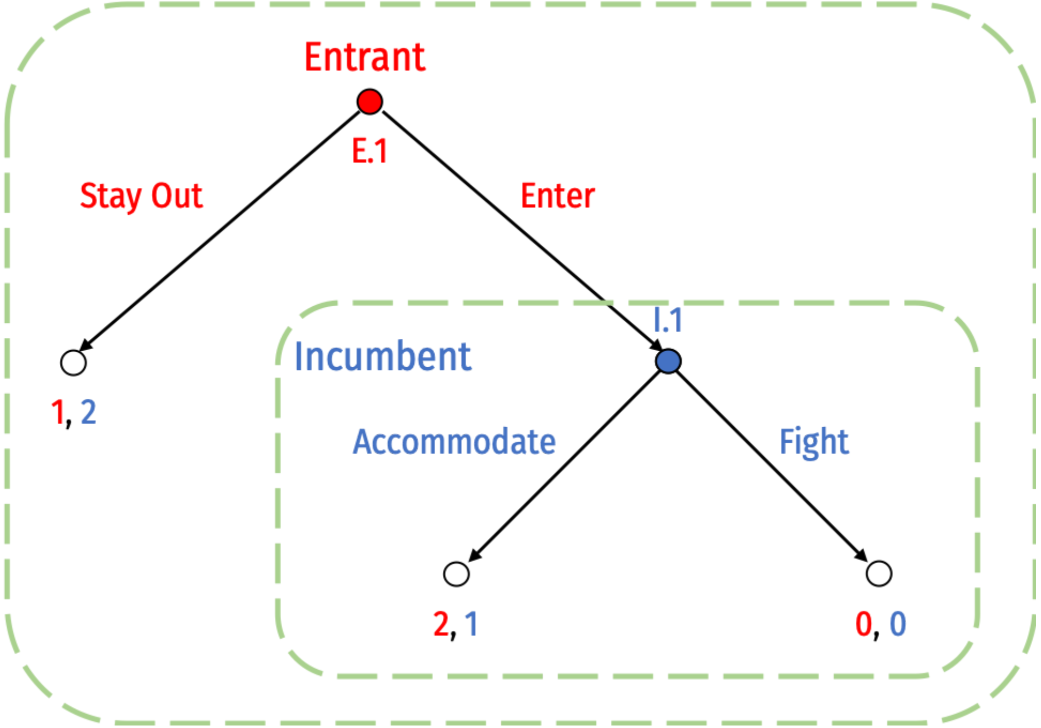
1. (**Enter**, **Accommodate**)
2. (**Stay Out**, **Fight**)

- But remember, we ignored the *sequential* nature of this game in normal form
  - Which Nash equilibrium is **sequentially rational?**

| | | Incumbent | |
|---|---|---|---|
| | | **Accommodate** | **Fight** |
| **Entrant** | **Enter** | 2, 1 | 0, 0 |
| | **Stay Out** | 1, 2 | 1, 2 |

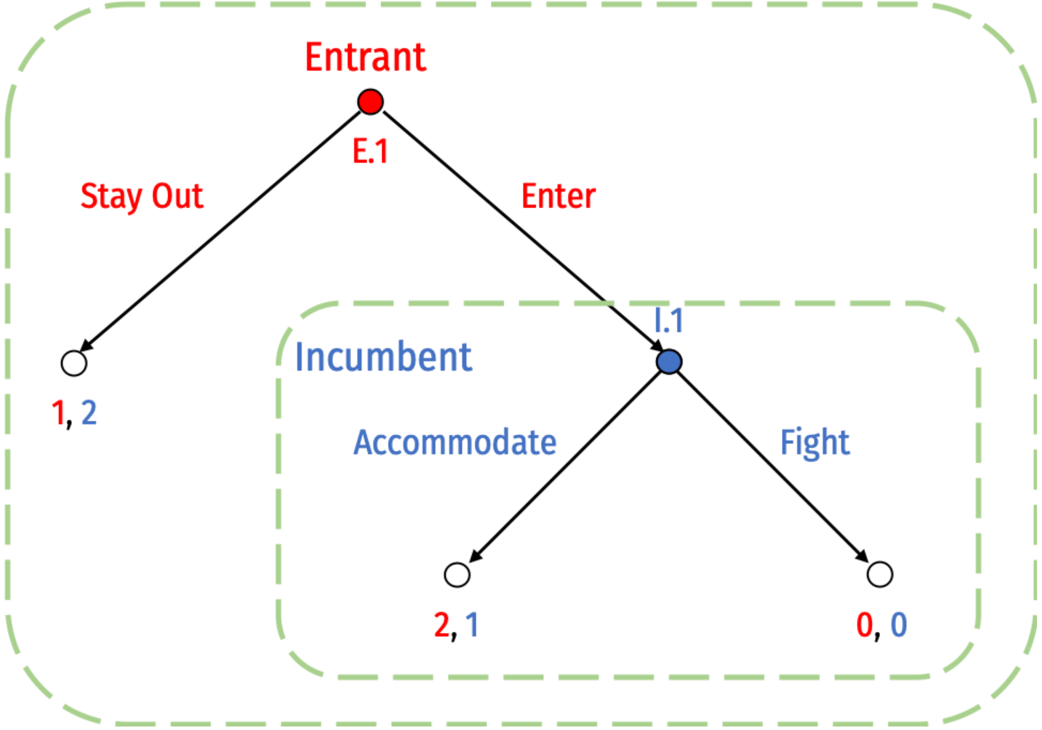
# Entry Game: Subgames

1. Subgame initiated at decision node **E.1** (i.e. the full game)
2. Subgame initiated at decision node **I.1**



# Entry Game: Subgame Perfect Nash Equilibrium

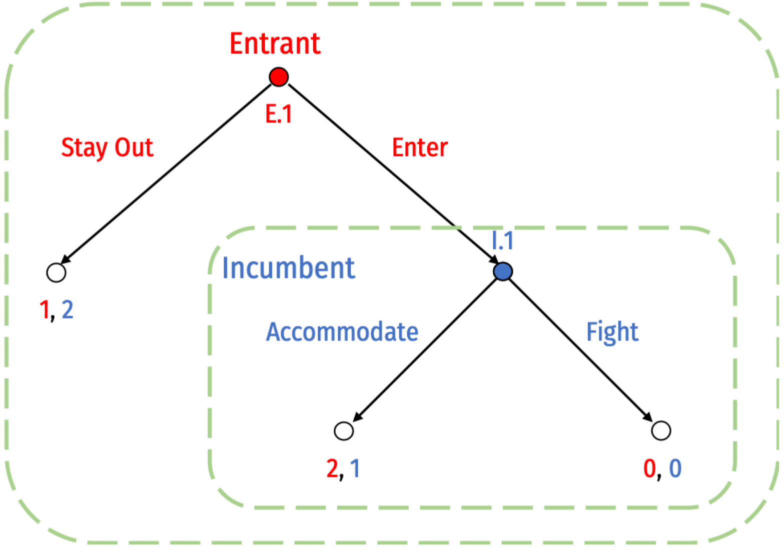
- Consider each subgame as a game itself and ignore the **"history"** of play that got a to that subgame
  - What is optimal to play in *that* subgame?
- Consider a set of strategies that is optimal for all players in *every* subgame it reaches
- That is a **subgame perfect Nash equilibrium**



# Entry Game: Subgame Perfect Nash Equilibrium

- Recall our two Nash Equilibria from normal form:

1. (**Enter**, **Accommodate**)
2. (**Stay Out**, **Fight**)



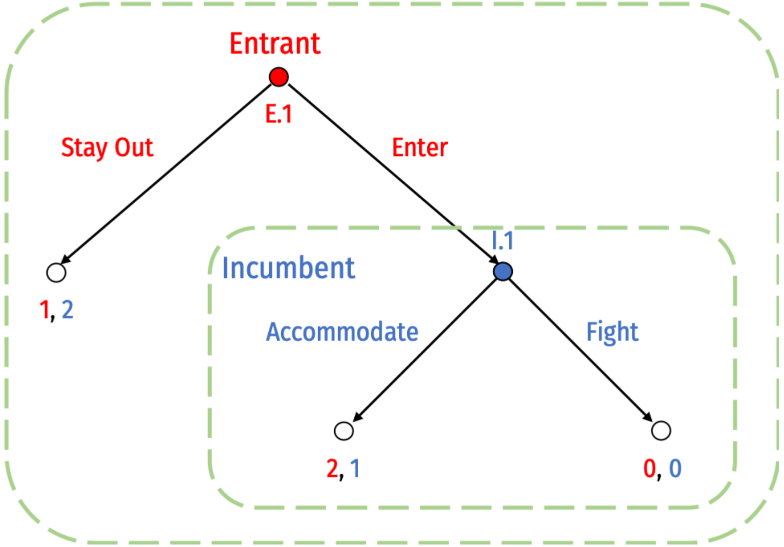

| | | Incumbent | |
|---|---|---|---|
| | | **Accommodate** | **Fight** |
| **Entrant** | **Enter** | 2, 1 | 0, 0 |
| | **Stay Out** | 1, 2 | 1, 2 |

# Entry Game: Subgame Perfect Nash Equilibrium

- Recall our two Nash Equilibria from normal form:

1. (**Enter**, **Accommodate**)
2. (**Stay Out**, **Fight**)

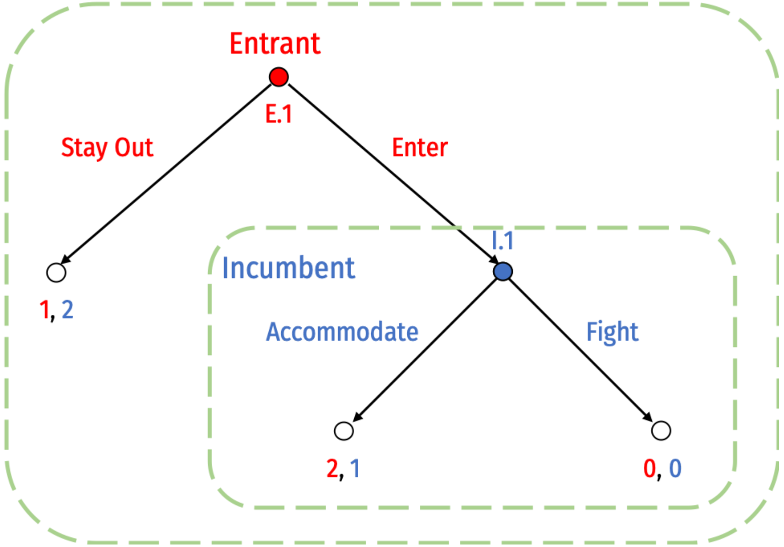
- Consider the second set of strategies, where **Incumbent** chooses to Fight at node I.1
- What if for some reason, **Incumbent** is playing this strategy, and **Entrant** unexpectedly plays Enter?




| | | Incumbent: Accommodate | Incumbent: Fight |
|---|---|---|---|
| **Entrant** | **Enter** | 2, 1 | 0, 0 |
| | **Stay Out** | 1, 2 | 1, 2 |

# Entry Game: Subgame Perfect Nash Equilibrium

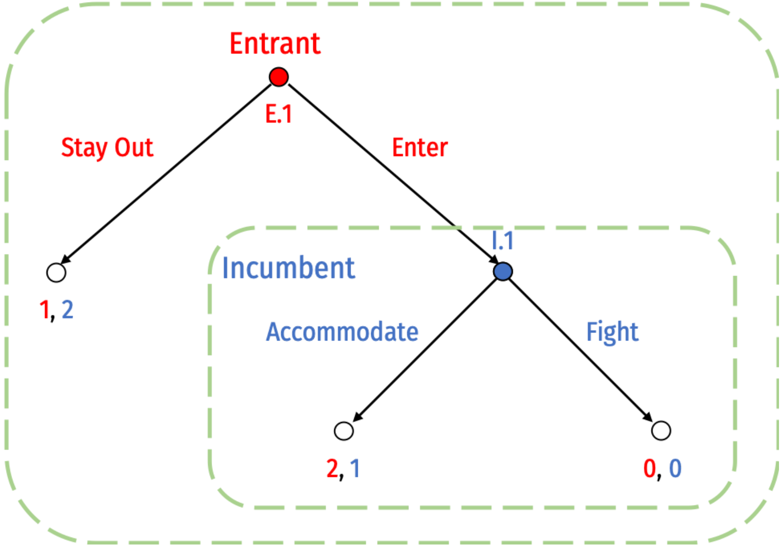
- It's **not rational** for **Incumbent** to play Fight if the game reaches I.1!
  - Would want to switch to Accommodate!
- **Incumbent** playing **Fight** at I.1 is **not a Nash Equilibrium in this subgame!**
- Thus, Nash Equilibrium (**Stay Out**, **Fight**) is **not sequentially rational**
  - It *is* still a Nash equilibrium!




| | | Incumbent: Accommodate | Incumbent: Fight |
|---|---|---|---|
| Entrant | Enter | 2, 1 | 0, 0 |
| Entrant | Stay Out | 1, 2 | 1, 2 |

# Entry Game: Subgame Perfect Nash Equilibrium

- Only (**Enter**, **Accommodate**) is a **Subgame Perfect Nash Equilibrium (SPNE)**
- These strategy profiles for each player constitute a Nash equilibrium in every possible subgame!
- Simple connection: rollback equilibrium is always SPNE!




| | | Incumbent: Accommodate | Incumbent: Fight |
|---|---|---|---|
| Entrant | Enter | 2, 1 | 0, 0 |
| Entrant | Stay Out | 1, 2 | 1, 2 |

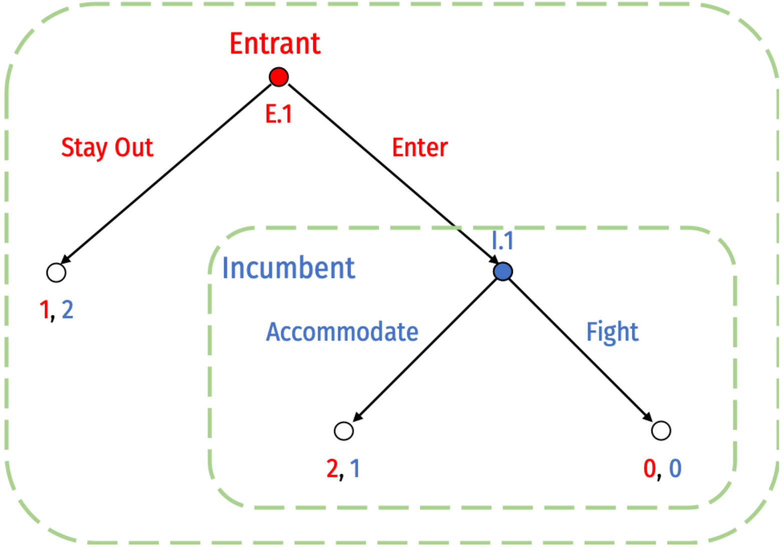
# Entry Game: SPNE and Credibility

- Suppose before the game started, **Incumbent** announced to **Entrant**

> "if you Enter, I will Fight!"

- This **threat** is **not credible** because playing Fight in response to Enter is not rational!

- The strategy is not Subgame Perfect!




| | | Incumbent | |
|---|---|---|---|
| | | **Accommodate** | **Fight** |
| **Entrant** | **Enter** | 2, 1 | 0, 0 |
| | **Stay Out** | 1, 2 | 1, 2 |

# Strategic Moves

# Strategic Moves AKA “Game Changers”

- So far, assumed rules of the game are fixed
- In many strategic situations, players have incentives to try to affect the rules of the game for their own benefit
  - Order, available strategies, payoffs, repetition
- A **strategic move** (“game changer”) is an action taken outside the rules an existing game by transforming it into a two-stage game
  - A strategic move is made in stage I (“pre-game” move)
  - A modified version of the original game is played in stage II

# Types of Strategic Moves

1. **Threats**: if other players don't choose your preferred move, you will play in a manner that will be bad for them (in second stage)
    - **Conditional** response to other players' actions
2. **Promises**: if other players choose your preferred move, you will play in a manner that will be good for them (in second stage)
    - **Conditional** response to other players' actions
3. **Commitments**: irreversibly limit your choice of action, **unconditional** on other players' actions

# Strategic Moves and Credibility

- Key: **threats and promises are often costly if you must carry them out against your own interest!**

- If a threat works and elicits the desired behavior in others, no need to carry it out

- If a promise elicits the desired behavior in others, cost of performing the promise

# Strategic Moves and Credibility

- For a strategic move to work, it must be:
  - observable to all players
  - irreversible so that it alters other players' expectations

- Other players must believe you will *actually do* in the second stage what you threaten/promise you will do during the first stage
  - **Credibility** of strategic moves open to question

# Strategic Moves and Credibility

- Your parents probably (tried to) used strategic moves on you
  - "No dessert unless you eat your vegetables"
  - "We'll buy you a new bike if you get a B GPA"
- You may have (rightly) questioned their credibility
  - Most parents *don't actually want* to punish or discipline their kids (it's painful *to the parents*)
  - (An empty) threat that changes their kid's behavior is great, but costly if it actually has to be carried out

# Non-Credibility AKA "Cheap Talk"

- "Talk is cheap"
  - Low cost to making promises/threats you don't intend to carry out
- Promises and threats **without commitment** will not change equilibrium behavior (with perfect information)
- If you try to bluff in poker, and your rivals know what cards you have, they will call your bluff

# Non-Credibility AKA "Cheap Talk"

- Promises or threats must be **incentive-compatible** to work
    - Threat/promise-maker must actually stand to benefit from performing the threat/promise or suffer from not performing it
- In game theory terms: strategy must be **subgame perfect**
- **Subgame perfection** rules out Nash equilibria relying upon non-credible threats and promises; keeps only behavior that is optimal under every circumstance!

# Credible Commitment

- Threats and promises can be **credible** with **commitment**
- A **commitment** changes the game in a way that forces you to carry out your promise or threat
  - tying your own hands makes you stronger!

# Credible Commitment

*Odysseus and the Sirens* by John William Waterhouse, Scene from Homer's *The Odyssey*

# Commitments

- A commitment is an action taken unconditional on other players' actions that limits your own actions
- If credible, tantamount to changing the order of the game at Stage II, so that the player making the commitment moves first
- Can change outcomes of following games, since it changes other players' expectations of the consequences of their own actions

# Simple Commitment Example in Chicken

- Take the game of Chicken
- Both players want to act tough from the beginning and project an image that they'll never back down, so the other player must
- But what makes a **credible commitment**?

**Column**

| Row | | Swerve | Straight |
|---|---|---|---|
| | **Swerve** | 0, 0 | -1, 1 |
| | **Straight** | 1, -1 | -2, -2 |

# Simple Commitment Example in Chicken

- Only a *visible* and *irreversible* action commits Row to going straight is **credible**
  - rip out steering wheel
  - tie the steering wheel
- Forces Column to Swerve

**Column**

| Row | | Swerve | Straight |
|---|---|---|---|
| | Straight | 1, -1 | -2, -2 |

# Simple Commitment Example in Chicken

# Simple Commitment Example in Chicken

# “Total Commitment” in *Dr. Strangelove*